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ON THE CONVOLUTION INVERSE OF DISCRETE SEQUENCES

PART I: Theory and estimation of the truncation error

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ABSTRACT: The aim of this paper is to prove the existence and unicity of the convolution inverse for a certain class of real functions of the discrete argument. The properties of these inverses, in particular the errors that arise from the truncation of the infinite sequences that represent them, are examined. In the second part of this paper subtitled "Application in solving convolution equations" (see [1]) the possibility of using convolution inverses for determining the solution to the Fredholm equations of the first kind is discussed.

1. INTRODUCTION

The problem of the inverse operation with respect to convolution plays an important role in numerous applications of science and technology. In terms of signal theory it is often called the deconvolution or inverse filtering (see for example [2, 3]). The purpose of the first part of this paper is to examine the existence and unicity of the convolution inverse of real functions of discrete integer argument. We deal with two-sided functions that represent noncausal physical phenomena and one-sided functions which represent causal (or anti-causal) phenomena. Special attention is paid to the estimation of the error that arises from the inevitable in practice truncation of the infinite sequence that represents the convolution inverse.

Now, we shall formulate some definitions concerning real functions of the discrete argument which will be used throughout this paper. Let $a = a(n)$, $b = b(n)$ be real functions of the discrete, integer argument n. If the series

$$\sum_{k=-\infty}^{\infty} a(k)b(n-k)$$

is convergent for every n, then the sum $c = c(n)$ of this series is called the convolution of functions a and b (see [4]). We write then

$$c(n) = a(n) * b(n) \text{ or } c = a * b$$

The function $\delta = \delta(n)$ defined as, we shall denote in the following form

$$a = \{a_r, a_{r+1}, \ldots\}$$

or

$$a = \{\ldots, a_{r-1}, a_r\}$$

respectively. A function

$$a = \{a_0, a_1, \ldots\} \text{ or } a = \{\ldots, a_{-1}, a_0\}$$

we call a right-sided or left-sided function, respectively. In signal theory these functions denote the causal and anti-causal signal, respectively. Finally, a function $a = a(n)$ which takes a finite (infinite) number of nonzero values we shall call a finite (infinite) function.

The organization of the paper is as follows. In Section 2 we consider the function

$$a = \{-a_{-p}, \ldots, -a_{-1}, 1, -a_1, \ldots, -a_q\},\ a_{-p}, a_q \neq 0 \tag{1.1}$$

(i.e. $a(n) = 0$ for $n \not\in \{-p, \ldots, 0, \ldots, q\}$) under the assumption

$$\sum_{\substack{i=-p \\ i\neq 0}}^{q} |a_i| = c < 1 \tag{1.2}$$

where p and q are arbitrarily fixed positive integers. Consequently, we formulate Theorem 1 concerning the existence and unicity of the convolution inverse of this function in space L_1. This theorem also gives the truncation error, which is important for practical applications. Th proof of Theorem 1 is given in Appendix A. Theorem 1 can be easily extended onto the function

$$a' = \{a'_{-p}, \ldots, a'_{-1}, a'_0, \ldots, a'_q\} \tag{1.3}$$

which satisfies the assumption

$$\sum_{\substack{i=-p \\ i\neq 0}}^{q} |a'_i| < |a'_0|,\ a'_0, a'_{-p}, a'_q \neq 0 \tag{1.4}$$

In Section 3 we consider an arbitrary right-sided function (causal signal)

$$a = \{a_0, -a_1, -a_2, \ldots\} \tag{1.5}$$

and a right-sided infinite function

$$a = \{1, -a_1, -a_2, \ldots\} \tag{1.6}$$

which satisfies the assumption

$$\sum_{i=1}^{\infty} |a_i| = c < 1 \tag{1.7}$$

We also consider a right-sided finite function (causal finite signal)

$$a = \{1, -a_1, \ldots, -a_q\} \tag{1.8}$$

which satisfies the assumption

$$\sum_{i=1}^{q} |a_i| = c < 1 , \qquad a_q \neq 0 \tag{1.9}$$

For the above functions we formulate and prove Theorem 2 concerning the existence and unicity of the convolution inverse and their certain estimations. Consequently, we discuss some properties of the convolution inverses of the functions (1.6) and (1.8) when in assumptions (1.7) and (1.9) we have $c = 1$ or $c > 1$.

In Section 4 we comment on results of the paper. In Appendix A the proof of Theorem 1 formulated in Section 2 is given. In Appendix B the existence of the convolution inverse, which does not belong to L_1, of the two-sided function

$$a = \{-a_{-1}, 1, -a_1\} , \qquad a_{-1}, a_1 \neq 0 , \qquad |a_{-1}| + |a_1| < 1 \tag{1.10}$$

is shown. Two lemmas concerning some properties of real function of discrete argument, which are used in Sections 2 and 3, are given in Appendix C.

2. CONVOLUTION INVERSE OF TWO-SIDED FUNCTIONS

Convolution inverse of the function (1.1) under assumption (1.2) has been discussed in [6, 7]. In [7] a certain algorithm for deriving convolution inverses in the Fourier transform domain and the estimate of the truncation error have been presented. In this section we shall consider the existence and unicity of the convolution inverse discussed in [6, 7] in more detail. We shall give a new and more useful estimation of the truncation error.

Consider the function

$$s = \{a_{-p}, \ldots, a_{-1}, 0, \ldots, a_1, \ldots, a_q\} \tag{2.1}$$

(viz. def. (1.1)) and define a sequence of functions of the discrete integer k

$$s_0, s_1, \ldots, s_n, \ldots \tag{2.2}$$

as follows

$$s_0 = \delta , \quad s_1 = s , \quad s_n = s_1 * s_{n-1} , \qquad n = 2, 3, \ldots \tag{2.3}$$

Using the sequence

$$b^{(n)} = s_0 + \cdots + s_n , \quad n = 0, 1, 2, \ldots \tag{2.4}$$

we formulate the following theorem.

THEOREM 1

If the assumption (1.2) is satisfied, then the sequence (2.4) is convergent in L_1, that is there exists a function $b = \{b(k)\} \in L_1$ such that

$$\|b^{(n)} - b\|_1 = \sum_{k=-\infty}^{\infty} |b^{(n)}(k) - b(k)| \rightarrow 0 \quad \text{for } n \rightarrow \infty$$

In addition, the function b is the unique convolution inverse of function (1.1) in L_1 and the following inequalities hold:

$$\|b\|_1 \leqslant (1-c)^{-1} \tag{2.5}$$

$$|b(0)| \leqslant (1-c)^{-1} \tag{2.6}$$

$$|b(nq+i)| \leqslant c^{n+1}(1-c)^{-1} , \qquad i = 1, \ldots, q; \quad n = 0, 1, 2, \ldots \tag{2.7}$$

$$|b(-np-i)| \leqslant c^{n+1}(1-c)^{-1} , \qquad i = 1, \ldots, p; \quad n = 0, 1, 2, \ldots \tag{2.8}$$

$$\frac{1}{p} \sum_{i=-np-1}^{-\infty} |b(i)|, \frac{1}{q} \sum_{i=nq+1}^{\infty} |b(i)| \leqslant \frac{c^{n+1}}{1-c} , \quad n = 0, 1, 2, \ldots \tag{2.9}$$

$$\sum_{i=-np}^{nq} |b(i) - b^{(n)}(i)| \leqslant c^{n+1}(1-c)^{-1} , \quad n = 0, 1, 2, \ldots \tag{2.10}$$

$$\|b - b^{(n)}\|_1 \leqslant (p+q+1)c^{n+1}(1-c)^{-1} , \quad n = 0, 1, 2, \ldots \tag{2.11}$$

The proof of Theorem 1 is given in Appendix A. It is interesting to note that for the function (1.1) which satisfies assumption (1.2) there may exist convolution inverses which do not belong to L_1. An example is shown in Appendix B.

Now, we briefly comment on the function $b^{(n)}$ given by eq. (2.4). In Appendix A there is shown that $b^{(n)}$ is a finite sequence defined by eq. (A.6). The convolution inverse b of a is an infinite sequence. Take into consideration the following truncation

$$\bar{b}^{(n)} = \{b(-np), \ldots, b(0), \ldots, b(nq)\} \tag{2.12}$$

of b. The function $b^{(n)}$ corresponds with $\bar{b}^{(n)}$, but $b^{(n)} \neq \bar{b}^{(n)}$. Relations (2.10), (2.12) and (A.6) yield the inequality

$$\|\bar{b}^{(n)} - b^{(n)}\|_1 \leqslant c^{n+1}(1-c)^{-1} , \quad n = 0, 1, 2, \ldots \tag{2.13}$$

Function $b^{(n)}$ is an approximation of $\bar{b}^{(n)}$ and b, and ineqs (2.13) and (2.11) give the estimation of the error for $b^{(n)}$ with respect to $\bar{b}^{(n)}$ and b, respectively. For sake of simplicity we can use the contracted notation for $b^{(n)}$, as follows

$$b^{(n)} = \delta + s + \cdots + s^n$$

where

$$s^2 = s * s, \ldots, s^n = s * s^{n-1}$$

Now, from [7] we shall recall formula (16) for the Fourier transform B of b (see also Appendix A), namely

$$B = [1 + S(j\omega)][1 + S^2(j\omega)][1 + S^4(j\omega)] \ldots = \prod_{i=0}^{\infty} [1 + S^{2^i}(j\omega)]$$

where $S(j\omega)$ is the Fourier transform of s. Hence, using the convolution theorem we can write

$$b^{(n)} = (\delta + s) * (\delta + s^2) * \cdots * (\delta + s^{2^k}), \quad n = 2^{k+1} - 1$$

where

$$s^2 = s * s, \quad s^4 = s^2 * s^2, \ldots, \quad s^{2^k} = s^{2^{k-1}} * s^{2^{k-1}}$$

Now consider the function (1.3) which satisfies assumption (1.4). Then, having denoted

$$-a_i'/a_0' = a_i, \qquad i = -p, \ldots, -1, 1, \ldots, q$$

we conclude that assumption (1.2) is satisfied and

$$a' = \{-a_0' a_{-p}, \ldots, -a_0' a_{-1}, a_0', -a_0' a_1, \ldots, -a_0' a_q\} = a_0' a$$

Hence, we have

$$(a')^{-1} = (a_0')^{-1} a^{-1}$$

The case of the two-sided function considered in this section represents the so-called non-causal function. The non-causal functions represent the phenomena that physically are not feasible and therefore studies on them might seem somewhat academic. In fact in all applications where the analysis of or processing need not be performed in real time (i.e. using computers) the use of non-causal representations of physical phenomena is allowed which means that studies on them are important.

The results of this section can be summarized as follows.

1° For the non-causal function (1.1) that satisfies assumption (1.2) there exists a unique inverse in L_1.

2° There may also exist other inverses not belonging to L_1.

It should be noted that the maximum element of the sequence a is placed at the origin. The assumption (1.2) is well known in the bibliography concerning the equalization of digital channels as the 'initial peak distortion less than one'. This assumption is specific for the existence of the optimum equalizer in L_1, called the zero-forcing-filter (see for example [8]).

3. CONVOLUTION INVERSE OF ONE-SIDED FUNCTIONS

We formulate and prove the following theorem.

THEOREM 2

The following assertions hold:
1° If $a_0 \neq 0$, then the function (1.5) has the unique right-sided convolution inverse

$$b = \{b_0, b_1, \ldots\} \tag{3.1}$$

where

$$b_0 = \frac{1}{a_0}\,, \quad b_n = \frac{a_n}{a_0}\, b_0 + \cdots + \frac{a_1}{a_0}\, b_{n-1}\,, \qquad n = 1, 2, \ldots \tag{3.2}$$

If $a_0 = 0$, then function (1.5) does not have right-sided convolution inverse.

2° If assumption (1.7) is satisfied, then the unique right-sided convolution inverse of function (1.6) is defined by the formulae

$$b = \{1, b_1, b_2, \ldots\} \tag{3.3}$$

$$b_n = a_n b_0 + \cdots + a_1 b_{n-1}\,, \qquad n = 1, 2, \ldots; b_0 = 1 \tag{3.4}$$

Moreover, $b \in L_1$ and

$$\|b\|_1 \leq (1 - c)^{-1} \tag{3.5}$$

3° If assumption (1.9) is satisfied, then for the right-sided convolution inverse of function (1.8) assertion 2° holds too. In addition formulae (3.4) have the following form

$$b_0 = 1\,, \quad b_1 = a_1 b_0, \ldots, \quad b_{q-1} = a_{q-1} b_0 + \cdots + a_1 b_{q-2} \tag{3.6}$$

$$b_n = a_q b_{n-q} + \cdots + a_1 b_{n-1}\,, \quad n = q, q+1, \ldots \tag{3.7}$$

and the following inequalities hold

$$|b_{nq+i}| \leq c^{n+1}(1 - c)^{-1}\,, \qquad i = 1, \ldots, q\,; \quad n = 0, 1, 2, \ldots, \tag{3.8}$$

$$\sum_{i=nq+1}^{\infty} |b_i| \leq q c^{n+1}(1 - c)^{-1}\,, \quad n = 0, 1, 2, \ldots \tag{3.9}$$

Proof

1° The convolution equation $a * b = \delta$ is equivalent to the following system of equations

$$a_0 b_0 = 1 \tag{3.10}$$

$$a_n b_0 + \cdots + a_1 b_{n-1} - a_0 b_n = 0\,, \quad n = 1, 2, \ldots \tag{3.11}$$

If $a_0 \neq 0$, then the above system of equations has exactly one solution given by formula (3.2). Hence, for function (1.5) there exists only one right-sided convolution inverse given by formulae (3.1), (3.2). If $a_0 = 0$, then the system of eqs (3.10) and (3.11) is inconsistent, i.e. there does not exist a right-sided convolution inverse for function (1.5).

2° The first part of assertion 2° results directly from assertion 1°. From formula (3.4) it follows that

$$|b_1| = |a_1|\,, \quad |b_2| \leq |a_2| + |a_1||b_1|$$

$$|b_3| \leq |a_3| + |a_2||b_1| + |a_1||b_2|$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|b_n| \leq |a_n| + |a_{n-1}||b_1| + |a_{n-2}||b_2| + \cdots + |a_1||b_{n-1}|$$

Denoting $d_n = |b_1| + \cdots + |b_n|$ and having summed up the above inequalities we obtain

$$d_n \leq c + c|b_1| + \cdots + c|b_{n-1}| \leq c + cd_n$$

Hence, we have $d_n \leq c(1-c)^{-1}$ which for $n \to \infty$ implies that

$$|b_1| + |b_2| + \cdots \leq c(1-c)^{-1}$$

Consequently, we have

$$\|b\|_1 = \sum_{n=0}^{\infty} |b_n| \leq (1-c)^{-1}$$

3° The first part of assertion 3° and formulae (3.6) and (3.7) result directly from assertion 2°. Now, we observe that the inverse (3.3) can be obtained in the same manner as in the proof of Theorem 1 for the two-sided function (1.1). Hence, ineqs (3.8) and (3.9) result directly from ineqs (2.7) and (2.9). This completes the proof of Theorem 2.

Using formulae (3.3), (3.6) and (3.7) the truncation of the infinite sequence (3.3) is defined as follows:

$$b^{(n)} = \{1, b_1, \ldots, b_{nq}\}\,, \quad n = 1, 2, \ldots \tag{3.12}$$

Then ineq. (3.9) has the form

$$\|b - b^{(n)}\|_1 \leq qc^{n+1}(1-c)^{-1}, \quad n = 1, 2, \ldots \tag{3.13}$$

This inequality gives an estimation of the error for the approximate convolution inverse $b^{(n)}$ in the sense of the L_1 norm.

Now, consider function (1.5) under the assumption

$$\sum_{i=1}^{\infty} |a_1| < |a_0| \tag{3.14}$$

and function

$$a = \{a_0, -a_1, \ldots, -a_q\} \tag{3.15}$$

under assumption

$$\sum_{i=1}^{q} |a_i| < |a_0|, \quad a_q \neq 0 \tag{3.16}$$

For the above cases, we obtain in the same manner as in Section 2, the following convolution inverses

$$b = a^{-1} = a_0^{-1}\{1, -a_1a_0^{-1}, -a_2a_0^{-1}, \ldots\}^{-1}$$

or

$$b = a^{-1} = a_0^{-1}\{1, -a_1a_0^{-1}, \ldots, -a_qa_0^{-1}\}^{-1}$$

Instead of ineq. (3.5) we have now

$$\|b\|_1 \leq |a_0|^{-1}(1-c)^{-1} \tag{3.17}$$

where

$$c = |a_0|^{-1} \sum_{n=1}^{\infty} |a_n| \tag{3.18}$$

for function (1.5) and

$$c = |a_0|^{-1} \sum_{n=1}^{q} |a_n| \tag{3.19}$$

for function (3.15). Ineqs (3.8) and (3.9) have the following form

$$|b_{nq+i}| \leq c^{n+1}|a_0|^{-1}(1-c)^{-1}, \quad i = 1, \ldots, q; \quad n = 0, 1, 2, \ldots \tag{3.20}$$

$$\sum_{i=nq+1}^{\infty} |b_i| \leq qc^{n+1}|a_0|^{-1}(1-c)^{-1}, \qquad n = 0, 1, 2, \ldots \tag{3.21}$$

Now, consider the function (1.6) under assumption

$$\sum_{n=1}^{\infty} |a_n| = c = 1 \tag{3.22}$$

The right-sided inverse of this function is given by formulae (3.3) and (3.4). Hence, by eq. (3.22), we obtain

$$b_0 = 1, \quad |b_1| = |a_1||b_0| < 1$$

$$|b_2| \leqslant |a_2||b_0|| + |a_1||b_1| < |a_2| + |a_1| \leqslant 1$$

$$|b_3| \leqslant |a_3||b_0| + |a_2||b_1| + |a_1||b_2| < |a_3| + |a_2| + |a_1| \leqslant 1$$

In this manner by induction one can prove that for the convolution inverse

$$b = \{1, b_1, b_2, \ldots\} \tag{3.23}$$

we have

$$|b_i| < 1, \qquad i = 1, 2, \ldots \tag{3.24}$$

This means that the inverse b is a bounded function but it need not necessary belong to L_1. For instance, the function

$$a = \{1, -\tfrac{1}{2}, -\tfrac{1}{4}, -\tfrac{1}{8}, \ldots\}$$

satisfies assumption (3.22). We have

$$b = a^{-1} = \{1, \tfrac{1}{2}, \tfrac{1}{2}, \ldots\}$$

but $b \not\in L_1$.

For function (1.6) which satisfies the assumption

$$\sum_{n=1}^{\infty} |a_n| = c > 1 \tag{3.25}$$

the convolution inverse (3.23) may be an unbounded function.

Now consider function (1.8) under assumption

$$\sum_{n=1}^{q} |a_n| = c = 1, \qquad a_q \neq 0 \tag{3.26}$$

If $a_n \neq 0$ for some $n \in \{1, \ldots, q-1\}$, then (as in the case of function (1.6)) the inverse (3.23) of function (1.8) also satisfies ineq. (3.24). If

$$a_n = 0, n = 1, \ldots, q-1; |a_q| = 1$$

then the inverse (3.23) of function (1.8) is given by the formula

$$b_0 = 1\,,\quad b_{nq} = (-\operatorname{sgn} a_q)^n\,,\quad n = 1, 2, \ldots;\qquad b_k = 0\,,\quad k \neq nq\,,\quad n = 1, 2, \ldots$$

In particular, we have

$$\{1, -1\}^{-1} = \{1, 1, 1, \ldots\}, \{1, 1\}^{-1} = \{1, -1, 1, -1, \ldots\}$$

Now, consider the function

$$a = \{1, -\tfrac{1}{2}, -\tfrac{1}{2}\}$$

Using formulae (3.6) and (3.7) we obtain

$$b_0 = 1\,,\quad b_1 = \tfrac{1}{2}\,,\quad b_n = \tfrac{1}{2}(b_{n-2} + b_{n-1})\,,\qquad n = 2, 3, \ldots$$

Hence, we have

$$b_n = b_{n-1} + (-\tfrac{1}{2})^n\,,\quad n = 1, 2, \ldots$$

which implies that

$$\lim_{n\to\infty} b_n = \tfrac{2}{3}$$

Also in this case $b \not\in L_1$.

If function (1.8) satisfies the assumption

$$\sum_{n=1}^{q} |a_n| = c > 1 \tag{3.27}$$

then the right-sided convolution inverse b of a may be an unbounded function. For instance, for the function

$$a = \{1, -1, -1\}$$

we have

$$b_0 = 1\,,\quad b_1 = 1\,,\quad b_n = b_{n-1} + b_{n-2}\,,\qquad n = 2, 3, \ldots$$

i.e. $b = \{1, 1, 2, 3, 5, \ldots\}$.

Finally, it should be emphasized that the determination of the left-sided convolution inverse of left-sided functions can be transformed into the above determination of the right-sided convolution inverse of the right-sided functions (see Lemma 1 in Appendix C).

In this section we have dealt with the right-sided functions of the discrete argument and their right-sided convolution inverses. Such functions are often referred to as causal functions which are used for the representation of the physically feasible phenomena. The results here obtained can be summarized as follows.

1° For the causal functions (1.5) and (3.15) satisfying assumptions (3.14) and (3.16), respectively there exist uniquely determined right-sided convolution inverses which belong to L_1.

2° For the causal functions (1.6) and (1.8) satisfying assumptions (3.22) and (3.26), respectively there exist uniquely determined bounded right-sided convolution inverses which need not belong to L_1.

3° For the causal functions (1.6) and (1.8) satisfying assumptions (3.25) and (3.27), respectively there exist uniquely determined right-sided convolution inverses which may be unbounded functions.

As in the case of two-sided functions considered in Section 2 the maximum element of the sequence a is placed at the origin. According to [9] the causal functions that satisfy assumption (3.16) are in fact minimum-phase sequences.

4. CONCLUSIONS

In this paper we have proven the existence and unicity of convolution inverses for a certain class of two-sided sequences (non-causal sequences) and one-sided sequences (causal or anti-causal sequences). We have also obtained approximate convolution inverses and the estimation of approximation error for these inverses in the sense of the L_1 norm (ineqs (2.11) and (3.13)).

It should be pointed out that the above approximate convolution inverses of two-sided sequences are not optimum in the sense of the L_1 error minimum. The optimum solution (in the L_1 sense) of the inverse problem for certain finite two-sided and one-sided sequences was published by R.W. Lucky in 1965 [10]. The linear system realizing this optimum solution is known as the ZFF (zero-forcing-filter). On the other hand the approximate convolution inverses of one-sided functions obtained in this paper are truncations of the exact convolution inverses (which are infinite one-sided sequences) and they are equal to those given in a form of ZFF. Consequently, for one-sided functions the above discussed approximate convolution inverses are optimum in L_1 sense.

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APPENDIX A

In this appendix we prove Theorem 1. For this purpose we shall use the Fourier transform definition for functions of discrete argument, e.g. [4]. Namely, for the function $a = a(n)$ its Fourier transform is given by the formula

$$A = A(j\omega) = \sum_{k=-\infty}^{\infty} a(k)\, e^{-j\omega k}, \quad \omega \in \mathbb{R} \tag{A.1}$$

provided that the above series is convergent for every $\omega \in \mathbb{R}$. Then we write $A = F(a)$. One can see that for $a \in L_1$ the series in formula (A.1) is absolutely and uniformly convergent for $\omega \in \mathbb{R}$. Hence, for $a \in L_1$ the transform $A = F(a)$ exists. In addition in this case we have

$$a(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(j\omega)\, e^{j\omega k}\, d\omega, \quad k = 0, \pm 1, \pm 2, \ldots \tag{A.2}$$

With reference to Lemma 2 (Appendix C) the relation $a, b \in L_1$ implies that $a * b \in L_1$ and consequently the transform $F(a * b)$ exists. One can prove that $F(a * b) = F(a)F(b)$ (convolution theorem).

Proof of Theorem 1

From the definition of convolution and formulae (2.1) and (2.3) it follows that

$$s_n = \{a_{-np}^{(n)}, \ldots, a_0^{(n)}, \ldots, a_{nq}^{(n)}\}, \quad n = 0, 1, 2, \ldots \tag{A.3}$$

where

$$a_0^{(0)} = 1, \quad a_0^{(1)} = 0 \tag{A.4}$$

$$a_i^{(1)} = a_i, \quad i = -p, \ldots, -1, 1, \ldots, q \tag{A.5}$$

Relations (2.4) and (A.3) imply that

$$b^{(n)} = \{b^{(n)}(-np), \ldots, b^{(n)}(0), \ldots, b^{(n)}(nq)\}, \quad n = 0, 1, 2, \ldots \tag{A.6}$$

We shall prove by induction that

$$\sum_{i=-np}^{nq} |a_i^{(n)}| \leq c^n, \quad n = 0, 1, 2, \ldots \tag{A.7}$$

Indeed, for $n = 0$ and $n = 1$ this inequality results from eqs (A.4) and (A.5) and assumption (1.2). Suppose that ineq. (A.7) is satisfied for $n = k$. Then, from the equality

$$\{a_{-(k+1)p}^{(k+1)}, \ldots, a_{(k+1)q}^{(k+1)}\} = a_{-p}\{a_{-kp}^{(k)}, \ldots, a_{kq}^{(k)}, 0, \ldots, 0\} + \cdots$$
$$+ a_q\{0, \ldots, 0, a_{-kp}^{(k)}, \ldots, a_{kq}^{(k)}\}$$

we have

$$\sum_{i=-(k+1)p}^{(k+1)q} |a_i^{(k+1)}| \leq |a_{-p}|c^k + \cdots + |a_q|c^k = c^{k+1}$$

i.e. ineq. (A.7) holds for $n = k + 1$.

Now consider the series

$$b(k) = \sum_{n=0}^{\infty} a_k^{(n)}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{A.8}$$

where

$$a_i^{(n)} = 0 \text{ for } n = 0, 1, 2, \ldots; i \notin \{-np, \ldots, nq\} \tag{A.9}$$

By ineq. (A.7) every series appearing in eq. (A.8) is absolutely convergent. Hence,

$$|b(k)| \leq \sum_{n=0}^{\infty} |a_k^{(n)}| < \infty, \quad k = 0, \pm 1, \pm 2, \ldots \tag{A.10}$$

Take any natural numbers m and r. Then, using relations (A.7) to (A.10) we find that

$$\sum_{k=-m}^{r} |b(k)| \leq \sum_{k=-m}^{r} \sum_{n=0}^{\infty} |a_k^{(n)}| = \sum_{n=0}^{\infty} \sum_{k=-m}^{r} |a_k^{(n)}| \leq \sum_{n=0}^{\infty} c^n = (1-c)^{-1}$$

Hence, for $m, r \to \infty$ we obtain

$$\sum_{k=-\infty}^{\infty} |b(k)| \leq (1-c)^{-1}$$

So, we have proven that the function $b = b(k)$ belongs to L_1 and satisfies ineq. (2.5). Ineq. (2.6) is an immediate consequence of ineq. (2.5). Ineqs (2.7) and (2.8) result from relations (A.7) to (A.9) inclusive. Consequently, ineqs (2.7) and (2.8) imply ineq. (2.9). Using relations (2.4), (A.3) to (A.6) and (A.9) we obtain

$$b^{(n)}(k) = \sum_{i=0}^{n} a_k^{(i)}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{A.11}$$

which implies that

$$b^{(n)}(k) = 0 \text{ for } k \notin \{-np, \ldots, 0, \ldots, nq\}, \quad n = 0, 1, 2, \ldots \tag{A.12}$$

By virtue of relations (A.8) and (A.11) we have

$$b(k) - b^{(n)}(k) = \sum_{i=n+1}^{\infty} a_k^{(i)}$$

Hence, taking into account relations (A.7) and (A.9) we obtain

$$\sum_{k=-np}^{nq} |b(k) - b^{(n)}(k)| \leqslant \sum_{k=-np}^{nq} \sum_{i=n+1}^{\infty} |a_k^{(i)}|$$

$$= \sum_{i=n+1}^{\infty} \sum_{k=-np}^{nq} |a_k^{(i)}| \leqslant \sum_{i=n+1}^{\infty} \sum_{k=-ip}^{iq} |a_k^{(i)}| \leqslant \sum_{i=n+1}^{\infty} c^i$$

which implies ineq. (2.10). Ineqs (2.9) and (2.10) and eq. (A.12) imply ineq. (2.11). Ineq. (2.11) yields the relation

$$\lim_{n\to\infty} \|b - b^{(n)}\|_1 = 0 \tag{A.13}$$

Hence, having denoted

$$B = B(j\omega) = F(b) , \qquad B^{(n)} = B^{(n)}(j\omega) = F(b^{(n)})$$

and using definition (A.1) of the Fourier transform we find that

$$\lim_{n\to\infty} |B^{(n)}(j\omega) - B(j\omega)| = 0 \tag{A.14}$$

uniformly with respect to $\omega \in \mathbb{R}$. From definitions (2.4), (2.3) we obtain

$$B^{(n)}(j\omega) = \sum_{i=0}^{n} S^i(j\omega) = [1 - S^{n+1}(j\omega)][1 - S(j\omega)]^{-1} , \quad n = 0, 1, 2, \ldots ; \omega \in \mathbb{R}$$

where $S = S(j\omega) = F(s)$ (see def. (2.1)). This result has already been given in [6, 7]. Hence, by the inequality

$$|S(j\omega)| \leqslant c < 1 , \qquad \omega \in \mathbb{R}$$

(following from definitions (A.1), (2.1) and assumption (1.2)), we obtain

$$\lim_{n\to\infty} B^{(n)}(j\omega) = [1 - S(j\omega)]^{-1}$$

uniformly with respect to $\omega \in \mathbb{R}$. Consequently, by relation (A.14), we have

$$B(j\omega) = [1 - S(j\omega)]^{-1} , \quad \omega \in \mathbb{R} \tag{A.15}$$

Now using definitions (2.1) and (1.1) we have

$$A(j\omega) = F(a) = F(\delta - s) = F(\delta) - F(s) = 1 - S(j\omega)$$

Hence, by eq. (A.15), we get

$$A(j\omega)B(j\omega) = 1 , \quad \omega \in \mathbb{R}$$

which implies that $F(a * b) = F(\delta)$. So, we have proven that $a * b = \delta$, i.e. the function b is the convolution inverse of the function a.

Now, suppose that there exists a convolution inverse $b' \in L_1$ of the function a.

Then $a * b = \delta$, which implies that

$$a * h = 0 \,, \quad h = b' - b$$

As $h \in L_1$, there exists the Fourier transform $H(j\omega) = F(h)$ and

$$A(j\omega)H(j\omega) = 0 \,, \quad \omega \in \mathbb{R}$$

Hence we have $H(j\omega) = 0$, $\omega \in \mathbb{R}$ which by formula (A.2) (with a and A replaced by h and H, respectively) implies that $h = 0$, i.e. $b' = b$. So we have proven the unicity of the convolution inverse of the function a in L_1, which completes the proof of Theorem 1.

APPENDIX B

The purpose of this appendix is to show that for the function (1.1) under assumption (1.2) there may exist a convolution inverse $h = h(n)$ which does not belong to L_1. For sake of simplicity we consider the function (1.10). Then the equation $a * h = \delta$ is equivalent to the following system of equations

$$a_{-1}h(n) - h(n-1) + a_1 h(n-2) = 0 \,, \quad n \neq 1 \tag{B.1}$$

$$a_{-1}h(1) - h(0) + a_1 h(-1) = -1 \tag{B.2}$$

Using the theory of recurrence equations [11] we solve the characteristic equation

$$W(r) = a_{-1}r^2 - r + a_1 = 0 \tag{B.3}$$

From the assumption

$$|a_{-1}| + |a_1| < 1 \,, \qquad a_{-1}, a_1 \neq 0 \tag{B.4}$$

it follows that $\Delta = 1 - 4a_{-1}a_1 > 0$. Hence, eq. (B.3) has two real roots

$$r_1 = (2a_{-1})^{-1}(1 - \sqrt{\Delta}) \,, \qquad r_2 = (2a_{-1})^{-1}(1 + \sqrt{\Delta}) \tag{B.5}$$

It follows from assumption (B.4) that $W(-1) > 0$, $W(1) < 0$. Hence, one of the roots r_1, r_2 belongs to the interval $(-1, 1)$ while the other one has modulus greater than 1. As from formula (B.5) we have $|r_1| < |r_2|$, hence

$$|r_1| < 1, |r_2| > 1 \tag{B.6}$$

The general solution of eq. (B.1) has the form

$$h(n) = \alpha r_1^n + \beta r_2^n \,, \quad n = 0, \pm 1, \pm 2, \ldots \tag{B.7}$$

where α, β are arbitrary real constants. The function (B.7) does not satisfy eq. (B.2) as for arbitrary α and β we have

$$a_{-1}h(1) - h(0) + a_1 h(-1) = 0$$

On the other hand one can easily find that the functions

$$h(n) = \begin{cases} \alpha r_1^n, & n = 0, 1, 2, \ldots \\ \alpha r_2^n, & n = -1, -2, \ldots \end{cases} \quad \text{with } \alpha = r_2(r_2 - 2a_1)^{-1} \tag{B.8}$$

$$h(n) = \begin{cases} \beta r_2^n, & n = 0, 1, 2, \ldots \\ \beta r_1^n, & n = -1, -2, \ldots \end{cases} \quad \text{with } \beta = r_1(r_1 - 2a_1)^{-1} \tag{B.9}$$

satisfy the system of eqs (B.1) and (B.2). Consequently, these functions are two different convolution inverses of the function (1.10). It follows from ineq. (B.6) that function (B.8) belongs to L_1 while the function (B.9) does not. It is clear that the method of inverse determination introduced in Section 2 gives the function (B.8).

APPENDIX C

In this appendix we formulate and prove two lemmas concerning properties of functions of discrete argument, which have been used previously.

Consider real functions of discrete integer argument

$$a = a(n), \quad b = b(n), \quad c = c(n)$$

and introduce the functions

$$a' = a'(n), \quad b' = b'(n), \quad c' = c'(n)$$

defined as follows

$$a'(n) = a(-n), \quad b'(n) = b(-n), \quad c'(n) = c(n)$$

In particular we have

$$\delta' = \delta'(n) = \delta(-n) = \delta(n) = \delta \tag{C.1}$$

LEMMA 1

The following assertions hold:

1° If $b = \alpha a$ (α being a real number), then $b' = \alpha a'$.
2° If $a + b = c$, then $a' + b' = c'$.
3° If there exists convolution $a * b = c$, then there exists convolution $a' * b'$ that equals c'.
4° If $b = a^{-1}$, then $b' = a'^{-1}$.
5° If there exists the transform $F(a) = A(j\omega)$, then there exists the transform $F(a') = A'(j\omega)$, where

$$A'(j\omega) = A(-j\omega) = A^*(j\omega)\,, \quad \omega \in \mathbb{R}$$

Assertions 1° to 3° and 5° can be easily proved with the aid of the definitions. Assertion 4° results from the definition of convolution inverse, eq. (C.1) and assertion 3°.

The next lemma concerns the convolution of two real functions of discrete integer argument.

LEMMA 2

If functions $a = a(n)$ and $b = b(n)$ belong to L_1, then the convolution a^*b exists, belongs to L_1 and there holds the inequality

$$\|a * b\|_1 \leqslant \|a\|_1 \|b\|_1 \tag{C.2}$$

Proof

Taking into consideration the relations

$$\|a\|_1 = \sum_{n=-\infty}^{\infty} |a(n)| < \infty, \; \|b\|_1 = \sum_{n=-\infty}^{\infty} |b(n)| < \infty$$

and Theorem I.11 of [12] we find that

$$\sum_{i,j} |a_i||b_j| = \|a\|_1 \|b\|_1 \tag{C.3}$$

Hence, by Theorem I.9 of [12], we have

$$d(n) = \sum_{i=-\infty}^{\infty} |a(i)||b(n-i)| < \infty \tag{C.4}$$

for any integer n and

$$\sum_{n=-\infty}^{\infty} d(n) = \|a\|_1 \|b\|_1 \tag{C.5}$$

In view of relation (C.4) the series

$$c(n) = \sum_{i=-\infty}^{\infty} a(i) b(n-i)$$

is convergent for any integer n, which implies the existence of the convolution $c = a * b$. Using eq. (C.5) and inequality $c(n) \leqslant d(n)$ we obtain the relation $a * b \in L_1$ and ineq. (C.2), which completes the proof.

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