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ON THE CONVOLUTION INVERSE OF DISCRETE SEQUENCES

PART II: Application in solving convolution equations

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ABSTRACT: In the first part of this paper subtitled 'Theory and estimation of the truncation error' (see [1]) we have examined the existence and unicity of the convolution inverse. In this part of the paper we discuss the application of convolution inverses for determining the solution to the Fredholm equation of the first kind. Particular attention is paid to the errors that arise from both the truncation of the infinite sequence that represents the inverse and the inaccuracy in input data.

1. INTRODUCTION

A problem of considerable importance in numerous areas of science and technology is the determination of the cause that produces a certain result, on the basis of the observation of the latter one. If the physical phenomenon that is investigated in such a way can be assumed to be a linear one, then the relationship between the cause, which is often called the input signal a, and the result called the output signal g is the equality

$$a * x = g \tag{1.1}$$

where x is the so called impulse response of the linear system that represents a certain physical phenomenon and the asterisk denotes convolution. Due to the commutativity of the convolution one can consider two possible cases:
- determination of x on the basis of the knowledge of a and g,
- determination of a on the basis of the knowledge of x and g.

Actually, this is a discretized version of the Fredholm equation of the first kind. In terms of signal theory the first of these cases is usually referred to as the non-parametric system identification, while the second one has no specific name. Perhaps the closest term which can be associated with it is the deconvolution or inverse filtering. It should be pointed out that from the viewpoint of mathematics and computational methods both cases are identical and the only difference is the physical interpretation of x and a. For this reason we shall be discussing the problem of the determination of x on the basis of the knowledge of a and g.

In the first part of this paper (see [1]) we have examined the existence and unicity of the convolution inverse. The purpose of this part is to apply the results of the first part to the determining the solution x of equation (1.1). We pay particular attention to the following two sources of errors:

1° the error that arises from the truncation of infinite sequence, that represents the convolution inverse of a;

2° the error that is due to the inaccuracy in the determination of g, which in practice is due to the inevitable noise and inaccuracy in measurements.

The analysis is performed for both two-sided (non-causal) or one-sided (causal or anti-causal) functions a and g. Throughout this part of the paper we use definition and notation concerning real functions of integer argument which were introduced in the first part.

2. TWO-SIDED CONVOLUTION EQUATIONS

In this section we shall use results of Section 2 of [1] for solving convolution eq. (1.1), where $a = a(k)$ and $g = g(k)$ are given two-sided functions of integer argument and $x = x(k)$ is an unknown function of integer argument. Notably, eq. (1.1) is the discretized version of the Fredholm integral equation of the first kind.

The goal of this section is to determine the solution of eq. (1.1) and the estimation of the truncation error that is inevitable in practical applications. We consider eq. (1.1) under the following assumption.

(2.I)

The function a is given by the formula

$$a = \{-a_{-p}, \ldots, -a_{-1}, 1, -a_1, \ldots, -a_q\}, \; a_{-p}, a_q \neq 0 \tag{2.1}$$

and satisfies the condition

$$\sum_{\substack{i=-p \\ i \neq 0}}^{q} |a_i| = c < 1 \tag{2.2}$$

where p and q are arbitrarily fixed positive integers. The function g is given by the formula

$$g = \{g(-p'), \ldots, g(0), \ldots, g(q')\}, \; g(p'), g(q') \neq 0\,, \tag{2.3}$$

where p' and q' are certain positive integers.

Let $b = b(k) \in L_1$ be the convolution inverse of function a determined by Theorem 1 of [1]. Then, by Lemma 2 (Appendix C, [1]), the function

$$x = x(k) = b(k) * g(k) \tag{2.4}$$

also belongs to L_1 and is a solution to eq. (1.1) (cf. [2], Section III.2.3). In addition, by ineq. (2.5) of [1], function x satisfies the inequality

$$\|x\|_1 \leq \|g\|_1 (1-c)^{-1} \tag{2.5}$$

In the same way as in the proof of Theorem 1 of [1] we conclude that the solution (2.4) of equation (1.1) is unique in the space L_1. On the other hand function a may possess a convolution inverse $b' = b'(k)$ that does not belong to L_1. In such a case the function

$$x' = x'(k) = b'(k) * g(k)$$

is also a solution to eq. (1.1) and $x' \not\in L_1$.

Now, replace the inverse b in the solution (2.4) by its truncated approximation $b^{(n)}$ given by formula (2.4) of [1], and g by $g + \varepsilon$, where

$$\varepsilon = \{\varepsilon(-p'), \ldots, \varepsilon(0), \ldots, \varepsilon(q')\} \tag{2.6}$$

Then formula (2.4) takes the form

$$x^{(n)} = x^{(n)}(k) = b^{(n)}(k) * [g(k) + \varepsilon(k)] \tag{2.7}$$

Function (2.7) is an approximate solution of eq. (1.1) while $\varepsilon = \varepsilon(k)$ can be interpreted as an inevitable error and/or noise that is always associated with the determination (measurement) of function g in practical applications. It follows from formulae (2.4) and (2.7) that

$$x(k) - x^{(n)}(k) = [b(k) - b^{(n)}(k)] * [g(k) + \varepsilon(k)] - b(k) * \varepsilon(k)$$

which by Lemma 2 of [1] implies that

$$\|x - x^{(n)}\|_1 \leq \|b - b^{(n)}\|_1 \|g + \varepsilon\|_1 + \|b\|_1 \|\varepsilon\|_1$$

Hence, by [1] (estimations (2.5) and (2.11)), we find that

$$\|x - x^{(n)}\|_1 \leq (1 - c)^{-1}[(p + q + 1)c^{n+1} \|g + \varepsilon\|_1 + \|\varepsilon\|_1] \tag{2.8}$$

The above inequality gives the estimation of error produced by replacing the exact solution x with the approximate solution $x^{(n)}$. By ineq. (2.8) and relations (2.1) and (2.2) we also have

$$\|g - a * x^{(n)}\|_1 = \|a * (x - x^{(n)})\|_1 \leq 2(1 - c)^{-1}[(p + q + 1)c^{n+1} \|g + \varepsilon\|_1 + \|\varepsilon\|_1] \tag{2.9}$$

Now we examine the existence of a finite solution

$$x = \{x(-p''), \ldots, x(0), \ldots, \ldots, x(q'')\} \tag{2.10}$$

of eq. (1.1) under assumption (2.I), where p'' and q'' are certain positive integers. It follows from relations (2.1) and (2.3) that such a solution may exist only in the case

$$p' > p,\ q' > q,\ p'' \geq p_1 = p' - p, \quad q'' \geq q_1 = q' - q \tag{2.11}$$

For the sake of simplicity denote

$$-a_i = a'_i\,, \quad i \neq 0\,, \quad a'_0 = 1$$

Then, by the definition of convolution, we obtain the following system of equations

$$a'_{-p}x(p'') = 0,\, a'_{-p+1}x(p'') + a'_p x(-p''+1) = 0, \ldots$$
$$\ldots + a'_{-p+1}x(-p_1 - 2) + a'_{-p}x(-p_1 - 1) = 0$$

Hence, we have

$$x(-p'') = \cdots = x(-p_1 - 1) = 0$$

In a similar way we conclude that

$$x(q'') = \cdots = x(q_1 + 1) = 0$$

Therefore, the solution (2.10) must have the form

$$x = \{x(-p_1), \ldots, x(0), \ldots, x(q_1)\}\,, \quad x(-p_1), x(q_1) \neq 0 \tag{2.12}$$

To derive the solution (2.12) we have only to solve the following system of equations

$$\sum_i a'_i x(n-i) = g(n)\,, \quad n = -p', \ldots, q' \tag{2.13}$$

This system may have at most one solution. Taking successive equations from system (2.13) for

$$n = q', q'-1, \ldots, q - p_1$$

we obtain the system of numbers

$$x(q_1), x(q_1 - 1), \ldots, x(-p_1)$$

If the above system satisfies the equations of system (2.13) for

$$n = q - p_1 - 1, \ldots, -p'$$

then function (2.12) is the unique finite solution of eq. (1.1) under assumptions (2.I) and (2.11). Otherwise, the system of eqs (2.13) is inconsistent and eq. (1.1) has no finite solution. Then, eq. (1.1) has the infinite solution given by formula (2.4) and this solution belongs to L_1. Of course, if for eq. (1.1) the finite solution (2.12) exists, then it is also consistent with formula (2.4).

Now, instead of eq. (1.1) consider the equation

$$a * x = g + \varepsilon \tag{2.14}$$

under assumption (2.I) with $p' > p$, $q' > q$, where ε is given by formula (2.6). Function $g + \varepsilon$ is an approximation of g and the parameter ε, as already mentioned, represents error arising from noise and inaccurate measurement of g. First we seek a finite solution (2.12) for eq. (2.14) in the manner described above. Suppose that $x^{(\varepsilon)}$ is such a solution. As

$$x^{(\varepsilon)} = b * (g + \varepsilon) \tag{2.15}$$

by relation (2.4) we obtain

$$\|x - x^{(\varepsilon)}\|_1 = \|b * \varepsilon\|_1 \leq \|b\|_1 \|\varepsilon\|_1 \leq (1 - c)^{-1} \|\varepsilon\|_1 \tag{2.16}$$

This inequality gives the estimation of the error that arises from replacing the exact solution x by the approximate one $x^{(\varepsilon)}$.

Suppose now that there does not exist a finite solution to eq. (2.14). Then we can determine the approximate, truncated solution $x^{(n)}$ with the aid of formula (2.7). At the same time the function $x^{(n)}$ is an approximate solution of eq. (1.1) and ineq. (2.8) gives the estimation of the truncation error.

3. ONE-SIDED CONVOLUTION EQUATIONS

The aim of this section is to derive the solution and the estimation of the truncation error for eq. (1.1) in the case where the functions a, g and x are the right-sided ones. This solution will be determined directly and with the aid of the right-sided convolution inverse examined in Section 3 of [1]. The existence of finite solutions is also discussed.

First assume that

$$a = \{a_0, -a_1, -a_2, \ldots\}, a_0 \neq 0 \tag{3.1}$$

$$g = \{g_0, g_1, g_2, \ldots\} . \tag{3.2}$$

If a right-sided function $x = \{x_n\}$ is a solution of eq. (1.1), then it is defined by the system of equations

$$a_0 x_0 = g_0, -a_n x_0 - \cdots - a_1 x_{n-1} + a_0 x_n = g_n , \quad n = 1, 2, \ldots$$

Consequently, we have

$$x_0 = \frac{g_n}{a_0} , \quad x_n = \frac{a_n}{a_0} x_0 + \cdots + \frac{a_1}{a_0} x_{n-1} + \frac{g_n}{a_0} , \quad n = 1, 2, \ldots \tag{3.3}$$

Hence, the function

$$x = \{x_0, x_1, x_2, \ldots\} \tag{3.4}$$

defined by formula (3.3) is the unique right-sided solution to eq. (1.1). On the other hand this solution is given by the formula

$$x = b * g \tag{3.5}$$

where b is the right-sided convolution inverse of a.

Now suppose that in addition

$$\sum_{n=1}^{\infty} |a_n| < |a_0| \tag{3.6}$$

and function (3.2) belongs to L_1. Then, by [1] (ineq. (3.17) and Lemma 2) and formula (3.5), we have

$$x \in L_1 \text{ and } \|x\|_1 \leq [|a_0|(1-c)]^{-1} \|g\|_1 \tag{3.7}$$

where

$$c = |a_0|^{-1} \sum_{n=1}^{\infty} |a_n| \tag{3.8}$$

Now suppose that

$$a = \{a_0, -a_1, \ldots, -a_q\}, a_0, a_q \neq 0 \tag{3.9}$$

(q being a positive integer) and

$$\sum_{n=1}^{q} |a_n| < |a_0| \tag{3.10}$$

and let g given by formula (3.2) belong to L_1. Then, the relation (3.7) holds with

$$c = |a_0|^{-1} \sum_{n=1}^{q} |a_n| \tag{3.11}$$

We shall consider eq. (1.1) in more detail under the following assumption.

(3.I)

Functions a and g are defined by the formulae

$$a = \{1, -a_1, \ldots, -a_q\}, a_q \neq 0 \tag{3.12}$$

$$g = \{g_0, g_1, \ldots, g_{q'}\} \tag{3.13}$$

where q and q' are certain positive integers. Moreover, we have

$$\sum_{i=1}^{q} |a_i| = c < 1 \tag{3.14}$$

Consider also eq. (2.14) with

$$\varepsilon = \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{q'}\} \tag{3.15}$$

Using formula (3.3) we have the following formulae for the solutions x and $x^{(\varepsilon)}$ of eqs (1.1) and (2.14), respectively:

$$x_0 = g_0 , \quad x_k = a_k x_0 + \cdots + a_1 x_{k-1} + g_k , \quad k = 1, 2, \ldots \tag{3.16}$$

$$x_0^{(\varepsilon)} = g_0 + \varepsilon_0 , \quad x_k^{(\varepsilon)} = a_k x_0^{(\varepsilon)} + \cdots + a_1 x_{k-1}^{(\varepsilon)} + g_k + \varepsilon_k , \quad k = 1, 2, \ldots \tag{3.17}$$

where

$$g_k = 0, \ \varepsilon_k = 0 \text{ for } k = q' + 1, \ q' + 2, \ldots$$
$$a_k = 0 \text{ for } k = q + 1, \ q + 2, \ldots$$

It follows from relation (3.7) that

$$x, x^{(\varepsilon)} \in L_1, \ \|x\|_1 \leq (1 - c)^{-1} \|g\|_1, \ \|x^{(\varepsilon)}\|_1 \leq (1 - c)^{-1} \|g + \varepsilon\|_1 \tag{3.18}$$

With the aid of relations (3.5) and (2.15) we find that ineq. (2.16) also holds in the considered case. The function $x^{(\varepsilon)}$ can be considered as an approximate solution of eq. (1.1). Then, ineq. (2.16) gives the approximation error arising from substituting x by $x^{(\varepsilon)}$. It is reminded that this error is due to noise and the inaccuracy of the measurement of g.

Now take the approximation

$$b^{(n)} = \{1, b_1, \ldots, b_n\} \tag{3.19}$$

(n being a positive integer) of the convolution inverse

$$b = \{1, b_1, b_2, \ldots\} \tag{3.20}$$

to function a. Then, the function

$$x^{(n)} = b^{(n)} * (g + \varepsilon) \tag{3.21}$$

is an approximate solution of eq. (2.14) and consequently eq. (1.1) too. We observe that

$$x^{(\varepsilon)}(k) = x^{(n)}(k) , \quad k = 0, 1, \ldots, n$$

Now, take $n = lq$, where l is a positive integer. Then, by ineq. (3.9) of [1], it follows that

$$\|b - b^{(n)}\|_1 = \sum_{i=lq+1}^{\infty} |b_i| \leq q(1 - c)^{-1} c^{l+1}$$

Consequently, we have

$$\|x^{(\varepsilon)} - x^{(n)}\|_1 \leq q c^{l+1} (1 - c)^{-1} \|g + \varepsilon\|_1 , \quad n = lq, \ l = 1, 2, \ldots, \tag{3.22}$$

which by virtue of ineq. (2.16) implies the following estimation

$$\|x - x^{(n)}\|_1 \leq (1-c)^{-1}[qc^{l+1}\|g+\varepsilon\|_1 + \|\varepsilon\|_1], \quad n = lq, l = 1, 2, \ldots \tag{3.23}$$

As in Section 2 we conclude that under assumption (3.I) a finite solution of eq. (1.1) may exist only if $q' > q$. If such a solution exists, it takes the form

$$x = \{x_0, x_1, \ldots, x_{q_1}\}, \quad q_1 = q' - q \tag{3.24}$$

To determine this solution the following system of equations must be solved

$$\sum_i a_i x_{n-i} = g_n, \quad n = 0, 1, \ldots, q' \tag{3.25}$$

where $a_0' = 1$, $a_i' = -a_i$, $i = 1, \ldots, q$. Having solved the subsystem of this system for $n = 0, 1, \ldots, q_1$ we obtain the system of numbers

$$x_0, x_1, \ldots, x_{q_1} \tag{3.26}$$

If this system of numbers satisfies the equations of system (3.25) for $n = q_1 + 1, \ldots, q'$, then function (3.24) is the solution of eq. (1.1). If the system (3.26) does not satisfy the equations of system (3.25) for $n = q_1 + 1, \ldots, q'$, then eq. (1.1) has no finite solution. Then, the infinite solution of eq. (1.1), that belongs to L_1, exists.

Now consider eq. (2.14) under assumption (3.I) with $q' > q$, where ε is given by formula (3.15). Having solved system (3.25) with g_n replaced by $g_n + \varepsilon_n$ (provided it has a solution) we obtain the following finite solution

$$x^{(\varepsilon)} = \{x_0^{(\varepsilon)}, \ldots, x_{q_1}^{(\varepsilon)}\}$$

of eq. (2.14). At the same time it is the approximate solution to eq. (1.1) and the ineq. (2.16) gives the estimation of the approximation error. If for eq. (2.14) a finite solution does not exist, then we take the approximate solution (3.21) for $n = lq$. It is also an approximate solution to eq. (1.1) and ineq. (3.23) gives the estimation of the approximation error.

Using Lemma 1 of [1] all results obtained above can be transformed to the case of convolution equations for left-sided (anti-causal) functions.

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