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CHEBYSHEV MINIMAX DECONVOLUTION FILTERING FOR A PAIR OF DISCRETE PULSES

Andrzej DYKA
Institute of Telecommunications, Technical University of Gdańsk, 80-952 Gdańsk, Poland

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ABSTRACT: This paper presents a closed form analytic solution for the impulse response of an optimum FIR deconvolution filter intended for a pair of discrete pulses of arbitrary amplitude and sign, subject to the minimisation of Chebyshev maximum norm for the approximation error. The tradeoff between the approximation error and the degradation of signal-to-noise ratio, is examined.

I. INTRODUCTION

For the discrete time domain the problem of ideal deconvolution filtering is usually defined in the following way. Given an input signal $\{s(n)\}$, determine the filter impulse response $\{h(n)\}$ such that,

$$\{s(n) * \{h(n)\} = \delta(n) \tag{1.1}$$

where

$$\delta(n) \triangleq \begin{Bmatrix} 1, & n=0 \\ 0, & n \neq 0 \end{Bmatrix}$$

and the asterisk denotes convolution.

Having assumed that both input signal $\{s(n)\}$ and impulse response $\{h(n)\}$ are sequences of finite length, one finds that the solution to $\{h(n)\}$ defined in such a way does not exist. Thus, the only way to approach the determination of $\{h(n)\}$ consists in using methods of approximation theory. The inevitable approximation error manifests itself in the appearance of additional pulses in the output signal, usually referred to as the sidelobes. Such an approximate solution to the deconvolution filtering problem can be represented by substituting $\delta(n)$ in the right hand side of (1.1) with the following output signal $\{g(n)\}$,

$$\{g(n)\} = g(0) \cdot \delta(n) + \sum_{\substack{i \\ i \neq 0}} g(i) \cdot \delta(n-i) \tag{1.2}$$

where $g(0) \cdot \delta(n)$ denotes the mainlobe, while the sum of $g(i) \cdot \delta(n-i)$, for $i \neq 0$, denotes the sidelobes. The appearance of sidelobes is considered to be an undesirable phenomenon in any system. To avoid the degradation of the system performance due to sidelobes one uses a special filter, which decreases their amplitude relative to the mainlobe amplitude. Notably, such a filter usually referred to as the Sidelobe Reduction Filter e.g. [3, 6] is a possible approximation of the ideal deconvolution filter defined by (1.1).

Consequently, the problem of the $\{h(n)\}$ determination can be formulated in terms of approximation theory as the minimisation of a certain measure of sidelobes, relative to the mainlobe amplitude. Unfortunately, the most commonly used mean square error norm as well as other 'integrating' norms do not enable us to control the sidelobes amplitude relative to the mainlobe amplitude, which is a serious disadvantage in numerous applications. The only definite solution to the problem of sidelobe amplitude control consists in using the minimisation of Chebyshev maximum norm for the approximation error.

This particular approach consists in the determination of the $\{h(n)\}$ impulse response subject to the following minimax criterion:

$$\inf_{\{h(n)\}\in H}\left[\frac{\max|g(i)|,\quad i\neq 0}{|g(0)|}\right] \tag{1.3}$$

where $\boldsymbol{H}$ denotes the set of all possible impulse responses $\{h(n)\}$. The value of (1.3) will be hereafter referred to as the minimum approximation error.

The direct application of this approach leads to the nonlinear minimax problem. Using the variable linearisation method presented in [4] this problem can be transformed into the linear one, leading to the Chebyshev solution of inconsistent linear equation systems, which unfortuantely requires lengthy iterative procedures [1]. However, in some specific form of input signal $\{s(n)\}$ the closed form analytic solution can be obtained [2–5, 8].

The aim of this paper is to present an analytic solution to the problem of deconvolution filtering for a pair of discrete pulses of arbitrary sign and amplitude, subject to the minimisation of Chebyshev maximum norm for the approximation error expressed by (1.3). This solution from now on will be referred to as the Chebyshev minimax solution or briefly the CMS.

The organization of the paper is as follows. Section II formulates the filtering problem and gives the solution, Section III discusses the filter performance namely, the tradeoff between the magnitude of the minimum approximation error and the degradation of signal-to-noise ratio, relative to the matched filter case, Section IV gives the summary of the results and specifies in brief possible applications.

II. FORMULATION AND SOLUTION TO THE FILTERING PROBLEM

Consider the input signal $\{s(n)\}$ as being a pair of impulses of arbitrary amplitude and sign,

$$\{s(n)\} = a\cdot\delta\left(n+\frac{1}{2}\right) + b\cdot\delta\left(n-\frac{1}{2}\right) \tag{2.1}$$

where $a\neq 0$, $b\neq 0$ are the arbitrary nonzero real numbers.

Now consider the impulse response $\{h_k(n)\}$ sought, whose length, understood here as the number of its samples, equals $2 \cdot k$.

$$\{h_k(n)\} = \sum_{i=-k+1}^{k} h_k\left(i - \frac{1}{2}\right) \cdot \delta\left(n - i + \frac{1}{2}\right), \quad k = 1, 2, \ldots, \tag{2.2}$$

where $h(i)$ denote the unknown elements of the impulse response to be determined.

With reference to (1.3) our objective here is to find $h_k(n)$ for $k = 1, 2, \ldots$ such, that the following infimum is attained,

$$\inf_{\{\hat{h}_k(n)\} \in \boldsymbol{H}_k} \left[\frac{\max|g(i)|, \quad i \neq 0}{|g(0)|} \right] \tag{2.3}$$

where $\boldsymbol{H}_k$ denotes the set of all functions $\{\hat{h}_k(n)\}$ which take the value 0 for $n = \pm(k + \frac{1}{2}),\ \pm(k + \frac{3}{2}),\ \pm(k + \frac{5}{2}), \ldots$.

The value of infimum (2.3) defines the minimum approximation error that can be achieved with the impulse response $\{h_k(n)\}$ sought, the latter being the element of $\boldsymbol{H}_k$. The specific number k which defines the length of the impulse response considered will be hereafter referred to as the filter order.

Using the theory and methods presented in [3–5] and [8] one can prove that the $\{h_k(n)\}$ that is sought takes the following form,

$$\{h_k(n)\} = \{h^M(n)\} * \{h^H_{k-1}(n)\} \tag{2.4}$$

where $\{h^M(n)\}$ denotes the impulse response of the filter matched to $\{s(n)\}$; and $\{h^H_{k-1}(n)\}$ denotes the impulse response of the minimax Sidelobe Reduction Filter—SRF for the Huffman sequence autocorrelation function type signals, [3].

Actually, the latter gives the CMS to the problem of the Chebyshev minimax deconvolution filtering for an even three element discrete signal, whose sidelobes amplitude does not exceed half of the mainlobe amplitude [3].

With reference to (2.1) the impulse response $\{h^M(n)\}$ of the filter matched to $\{s(n)\}$ can be explicitly expressed as:

$$\{h^M(n)\} = b \cdot \delta\left(n + \frac{1}{2}\right) + a \cdot \delta\left(n - \frac{1}{2}\right). \tag{2.5}$$

Consequently, the response of this filter to $\{s(n)\}$ is the autocorrelation function of the latter given as,

$$\{r(n)\} = \delta(n) + \frac{\alpha}{2} \cdot [\delta(n-1) + \delta(n+1)]. \tag{2.6}$$

where

$$\alpha \equiv \frac{2ab}{a^2 + b^2}.$$

The coefficient α can be interpreted as a kind of equality coefficient between the input pulses amplitude a and b. It takes the value of 1 or -1 for $a = b$ or $a = -b$, respectively. For $|a| \neq |b|$ the value of $|\alpha| < 1$. Hence, it follows that

$$|\alpha| \leq 1. \tag{2.7}$$

Table 1

k	$\{h_k(n)\}$ $a=b=1$, even pair	$\{h_k(n)\}$ $a=1, b=-1$, odd pair
1	$\{1, 1\}$	$\{-1, 1\}$
2	$\{-1, 2, 2, -1\}$	$\{-1, -2, 2, 1\}$
3	$\{1, -2, 3, 3, -2, 1\}$	$\{-1, -2, -3, 3, 2, 1\}$
4	$\{-1, 2, -3, 4, 4, -3, 2, -1\}$	$\{-1, -2, -3, -4, 4, 3, 2, 1\}$
and so forth		

It has been proved in [3] that if (2.7) is satisfied then the unique solution to $\{h^H_{k-1}(n)\}$, given in a closed analytic form, exists for any value of k. It has also been proved there, that the filter corresponding to the $\{h^H_{k-1}\}$ can be shown as a parallel combination of finite-length zero forcing filters, ZFF's, e.g., [7]. Hence, this and (2.5) imply the existence of the unique solution to $\{h_k(n)\}$ expressed by (2.4).

Consequently, by virtue of (2.4) and (2.5) the impulse response $\{h_k(n)\}$ that is sought can be presented as:

$$\{h_k(n)\} = b \cdot \{h^H_{k-1}(n)\} + a \cdot \{h^H_{k-1}(n-1)\} . \tag{2.8}$$

A specific form of input signal $\{s(n)\}$ is for $|b/a| = 1$, i.e. for $|\alpha| = 1$, (2.7). Particularly, for $a = b = 1$ and $a = 1$, $b = -1$ input signal $\{s(n)\}$ takes the form of an even pair of impulses with the Fourier transform being the cosine function and an odd pair of impulses with the Fourier transform being the sine function, respectively. In these cases the impulse response $\{h_k(n)\}$ of the deconvolution filter discussed can be derived in a recurrence manner, as shown in Table 1.

III. PERFORMANCE

In order to evaluate the performance of the filtering here presented, we shall recall the performance indices used in [3], namely the peak-sidelobe ratio PSLR and the output signal-to-noise ratio SNR_0. The first of them is in fact the minimum approximation error (2.3) expressed in decibels, i.e.,

$$\mathrm{PSLR} \triangleq 20 \log \left[\frac{\max|g(i)|, \quad i \neq 0}{|g(0)|} \right], \quad [\mathrm{dB}] \tag{3.1}$$

The output signal-to-noise ratio has been defined as,

$$\mathrm{SNR}_0 \triangleq 10 \log \frac{g^2(0)}{\sum_{i=-k+1}^{k} h^2\left(i - \frac{1}{2}\right)}, \quad [\mathrm{dB}] \tag{3.2}$$

provided that $a^2 + b^2 = 1$.

The latter condition secures that the SNR_0 defined by (3.2) is the measure of the output signal-to-noise ratio degradation, relative to the matched filter case, e.g., [6].

Matched filter which yields the maximum of output signal-to-noise ratio is considered thus as a reference.

From [3] one can derive,

$$\mathrm{PSLR} = -20\log\left[2\sum_{i=1}^{k}|T_i(1/\alpha)|\right], \quad [\mathrm{dB}] \tag{3.3}$$

where $T_i(\ )$ denotes the ith order Chebyshev polynomial of the first kind.

For the case of $|\alpha| = 1$, i.e., for the even or odd pair of impulses, formula (3.3) can be reduced to

$$\mathrm{PSLR} = -20\log(2k)\,. \tag{3.4}$$

Notably, the signal-to-noise ratio varies with increasing impulse response length, the latter expressed by the filter order k. In general, an analytic expression for the SNR_0 vs. k is rather complicated, and the use of the definition (3.2) was found to be more convenient in practice. However, in the specific case of $|\alpha| = 1$, i.e., for the pair of impulses of equal amplitude, the expression for SNR_0 takes a simple form as follows,

$$\mathrm{SNR}_0 = 10\log\frac{6(k+1)}{(k+2)(2k+3)}\,, \quad k = 2, 3, \ldots\,. \tag{3.5}$$

The value of $\lim \mathrm{SNR}_0$, for $k \to \infty$, shows the tendency of the SNR_0 for very 'long' filters. Using the theory presented in [8] after some routine though tedious transformations one obtains the following:

$$\lim_{k\to\infty} \mathrm{SNR}_0 = 10\log(1-\alpha^2)^{1/2}\,, \quad \text{for } |\alpha| < 1\,. \tag{3.6}$$

By (3.5) it follows that

$$\lim_{k\to\infty} \mathrm{SNR}_0 = -\infty\,, \quad \text{for } |\alpha| = 1\,. \tag{3.7}$$

The above examinations show the tradeoff between the minimum of the approximation error, expressed by the PSLR, and the degradation of output signal-to-noise ratio, given by SNR_0. For the input signal being a pair of impulses of arbitrary unequal amplitude the magnitude of the minimum approximation error decreases monotonically with increasing filter length, while the SNR_0 converges asymptotically to its finite value lower bound. The lower the value of $|\alpha|$ the quicker the convergence of SNR_0 to the asymptote (3.6) vs. filter length k. For the input signal being the even or odd pair of impulses the situation is somewhat different. The magnitude of the minimum approximation error also decreases monotonically with increasing filter length and the SNR_0 has no finite value lower bound and decreases monotonically too. Formulae (3.4) and (3.5) show that the rate of decrease of both PSLR and SNR_0 against the filter length k are of the same order. The behaviour of the PSLR and SNR_0 vs. $|\alpha|$ for $k = 1, \ldots, 5$ are shown in Figs. 1 and 2, respectively.

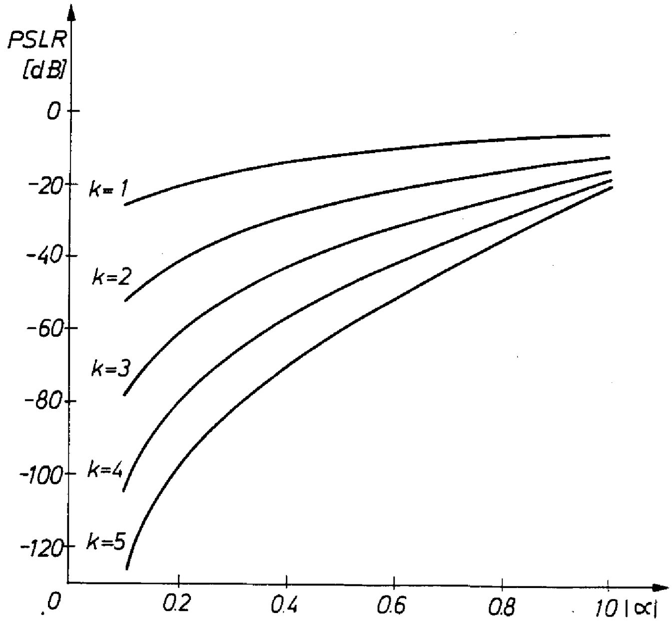

Figure 1: Minimum approximation error (PSLR) vs. $|\alpha|$ for the impulse response length $k = 1, \ldots, 5$.

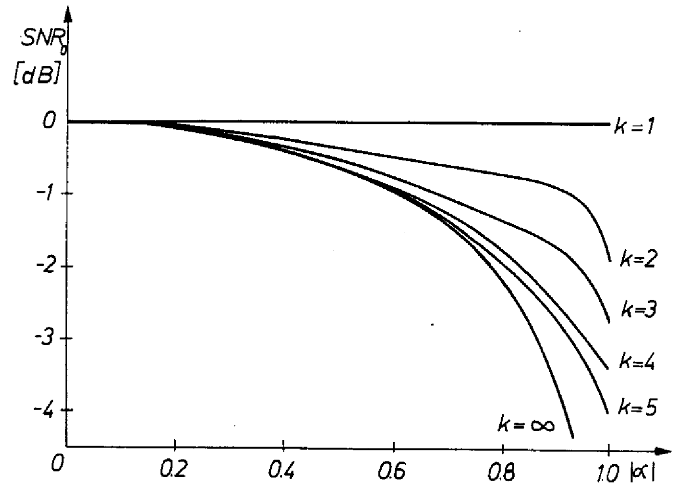

Figure 2: Output signal-to-noise ratio (SNR_0) vs. $|\alpha|$ for the impulse response length $k = 1, \ldots, 5$.

4. CONCLUDING REMARKS

The degradation of the signal-to-noise ratio SNR_0, relative to the matched filter can be considered as a price that is to be paid for decreasing the minimum approximation error expressed by the PSLR. The tradeoff between the PSLR and the SNR_0 is quite favourable for an input signal consisting of two pulses of unequal amplitude, i.e. for $|\alpha| < 1$. The larger the difference in the amplitude of input pulses, (i.e., the lower the $|\alpha|$), the lower the maximum price expressed by the lim SNR_0 for $k \to \infty$ to be paid for decreasing the approximation error. In the limiting worst case of input pulses of equal amplitude, i.e., for $|\alpha| = 1$, the tradeoff between the PSLR and SNR_0 is well balanced—both of them are linearly dependent on the filter length k.

Although the idea of studies on filtering such simple signals may seem somewhat academic, a number of applications for the solution here obtained can be pointed out. For instance, in radar, (sonar) systems the Fast Time Constant, FTC is commonly used to improve the target detectability in clutter, (reverberation), e.g., [6]. The filtering here presented can be used to remove the ambiguity in the FTC output signal, giving thus farther improvement in the target detectability. Another possibility consists in using the solution here obtained as an FIR integrator which performs the inverse operation to the differentiation, the latter represented by the odd pair of impulses, (Table 1). And the last but not least—for a given filter length k, the CMS solution here presented gives the minimum of the approximation error in the sense of the Chebyshev minimax norm that can be obtained with any linear filter.

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