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A NONITERATIVE ALGORITHM FOR DECONVOLUTION-INVERSE FILTERING USING THE CHEBYSHEV MINIMAX NORM FOR THE APPROXIMATION ERROR

Part II: Performance

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ABSTRACT: The aim of this paper is to show that in the case of even input signals with sidelobes of equal amplitude and arbitrary sign the D-algorithm introduced in the first part of this paper subtitled "Theory", may give a solution which is equal to that with the Chebyshev minimax norm for the approximation error. It is proved that, with some restrictions, in the case of two, four, and six sidelobe even input signals, the algorithm discussed gives exactly the Chebyshev minimax solution—CMS. Also, properties of the algorithm in the case of more general input signal are discussed.

INTRODUCTION

In the first part of this paper subtitled "Theory" [1], the D-algorithm, also presented in brief in [2], giving solution to deconvolution filtering problems, was introduced and its potential properties were examined. Particularly, it was found that in the case of even input signals with sidelobes of equal amplitude and arbitrary signs the algorithm discussed may give the Chebyshev minimax norm solution, which was referred to as the CMS. We have also shown that in a more general case of even input signal, when the CMS can not be achieved, the D-algorithm solution provides a good prompt for determining an "appropriate subsystem" of $n + 1$ linear equations of n unknowns, which in turn gives the CMS, directly. This prompt has been formulated in the form of the criterion (K).

With reference to Section 2 in [1] we should remember that the minimax solution of the $p + q$ inconsistent linear equations of p unknowns is a solution to the appropriate subsystem of $p + 1$ equations. Another property states, that in the $p + 1$ minimax problem with p unknowns, $p + 1$ equal in magnitude errors in the solution is to be expected [3]. In view of the above and condition (C_2), (Section 4 of [1]), the two following cases should be distinguished.

(1) Input signal $s(n)$ has two sidelobes of equal amplitude and arbitrary sign. In this case, using the D-algorithm we can always satisfy condition (C_2), necessary to obtain the CMS.

(2) Input signal $s(n)$ has more than two sidelobes of equal amplitude and arbitrary sign. In this case, using the D-algorithm, we can secure that the condition (C_2) is met only for $l = 1$, $p = q$ (eq. (3.1) in [1]).

These are necessary conditions, to ensure that the D-algorithm provides the CMS. In particular, the above conditions allow the input signal to be an even function. As for the practical reasons mentioned in Section 1 of [1] even input signals are of particular relevance, we shall restrict our considerations to such a class of signals. Consequently, we shall formulate sufficient conditions, which secure the CMS in the case of input signal with two, four, and six sidelobes, respectively. We shall also briefly discuss the performance of the algorithm in a more general case of input signal.

The organization of the paper is as follows. Sections 1, 2 and 3 formulate necessary and sufficient conditions which secure the CMS in the case of two, four, and six sidelobe even input signal, respectively, Section 4 is the discussion on the results, Appendices A, B and C give derivations of the formulae and proofs of the theorems formulated in Sections 1, 2 and 3, respectively.

1. CASE OF INPUT SIGNAL WITH TWO SIDELOBES

With reference to (eq. (1.1) in [1]) this case corresponds to the following assumption,

$$q = 1 \tag{1.1}$$

which implies that $a_1 = a$, and $a_i = 0$, for $i = 2, 3, \ldots$. Detailed analysis for this case has been presented in [4, 5]. Particularly, for $0 < |a| \leq 0.5$ it was proved that the D-algorithm gives the CMS exactly for any length of the impulse response i.e. for $l = 1, 2, \ldots$, (eq. (3.1) in [1]). The CMS can be also obtained for $|a| > 0.5$, however in this case certain values of a, depending on the impulse response length, must be excluded. Basically, the larger the length l of the impulse response, the more specific values of a are to be excluded. For instance,

$$\left.\begin{array}{llll} \text{for } l = 1, & a \neq \pm\sqrt{2}/2, & a \neq \pm 1, & \\ \text{for } l = 2, & a \neq \pm\sqrt{3}/3, & a \neq \pm\sqrt{2}/2, & a \neq \pm 1 . \end{array}\right\} \tag{1.2}$$

Although on the basis of the theory presented in [5] it is possible to derive all values of a to be excluded for any impulse response length l, it was found, that for larger values of l this may lead to rather lengthy calculations. It was found to be more convenient to use certain criterion conditions for any specific value of a and l. If these conditions are satisfied, then the particular value of a is allowed. Derivation of these conditions as well as an illustrative example are shown in Appendix A.

2. CASE OF INPUT SIGNAL WITH FOUR SIDELOBES

Consider the problem (M), assuming that for eqs. (1.1) and (1.2) in [1] we have the following case

$$q = p = 2, \quad a > 0, \quad a_i = a \circ e_i, \quad e_i = \pm 1, \quad \text{for } i = 1, 2 . \tag{2.1}$$

Now, designate

$$F_1 = [(0, 1) \cup (3, \infty)] \setminus \{0.6789631\}, \tag{2.2}$$

$$F_2 = (0, \infty) \setminus \left\{ \frac{-1+\sqrt{5}}{2};\ 0.3406654 \right\} \tag{2.3}$$

and assume that,

$$a \in F_1 \quad \text{if } e_2 = 1, \tag{2.4}$$

$$a \in F_2 \quad \text{if } e_2 = -1. \tag{2.5}$$

Then, using the D-algorithm, by eqs. (3.16) and (3.17) in [1] we obtain the following eqs.

$$\hat{x}_1 = \frac{e_1 \circ (ae_2 - 1)}{1 + ae_2 - a^2 + a \circ |2 - ae_2| \circ \operatorname{sgn} B}, \tag{2.6}$$

$$\hat{x}_2 = \frac{-ae_2}{1 + ae_2 - a^2 + a \circ |2 - ae_2| \circ \operatorname{sgn} B}, \tag{2.7}$$

$$H = \frac{a^2 \circ |2e_2 - a|}{a \circ |2e_2 - a| + |B|}, \tag{2.8}$$

where

$$B = 1 + ae_2 - 5a^2 + 2e_2 a^3. \tag{2.9}$$

THEOREM 1: Having assumed (2.1), the system $\hat{x} = (\hat{x}_1, \hat{x}_2)$, given by eqs. (2.6) and (2.7), is the unique solution to the problem (M) for the assumption (2.5) and in the following case:

$$a \in (0, 1) \setminus \{0.6789631\} \quad \text{for } e_2 = 1. \tag{2.10}$$

The value of H, in eq. (2.8) equals the infimum (1.4) in [1].

The proof of this theorem is given in Appendix B.

3. CASE OF INPUT SIGNAL WITH SIX SIDELOBES

Consider the problem (M) assuming that for eqs. (1.1) and (1.2) in [1] we have the following case

$$q = p = 3, \quad a > 0, \quad a_i = ae_i, \quad e_i = \pm 1, \quad \text{for } i = 1, 2, 3. \tag{3.1}$$

Now, designate

$$G_1 = (0, 1/2) \cup (2/3, 1) \cup \left(\frac{7+\sqrt{33}}{2}, \infty \right), \tag{3.2}$$

$$G_2 = (0, \infty)\setminus\{0.2234578; 0.3142734\}, \tag{3.3}$$

$$G_3 = (0, \infty)\setminus\{0.3237679; 0.8019377\}, \tag{3.4}$$

$$G_4 = \left[\left(0, \frac{4-\sqrt{7}}{3}\right) \cup (3, \infty)\right]\setminus\{0.4237189\}, \tag{3.5}$$

and assume the following conditions,

$$a \in G_1 \quad \text{if } e_1e_3 = 1, \text{ and } e_2 = 1, \tag{3.6}$$

$$a \in G_2 \quad \text{if } e_1e_3 = 1, \text{ and } e_2 = -1, \tag{3.7}$$

$$a \in G_3 \quad \text{if } e_1e_3 = -1, \text{ and } e_2 = 1, \tag{3.8}$$

$$a \in G_4 \quad \text{if } e_1e_3 = -1, \text{ and } e_2 = -1. \tag{3.9}$$

Then, using the D-algorithm, by eqs. (3.16) and (3.17) in [1] we obtain the following eqs.

$$\hat{x}_n = \frac{ax'_n}{a + \varepsilon_1 \circ \operatorname{sgn}(W_0 \circ W)}, \quad \text{for } n = 1, 2, 3. \tag{3.10}$$

$$H = \frac{a\varepsilon_1}{\varepsilon_1 + a \circ |W_0/W|} \tag{3.11}$$

where

$$x'_1 = -a \circ [e_1 - a \circ (2e_2e_3 + e_1e_2) + a^2(e_1 + e_3)] \circ W^{-1}, \tag{3.12}$$

$$x'_2 = -a \circ (e_2 - 2ae_1e_3 + a^2e_2) \circ W^{-1}, \tag{3.13}$$

$$x'_3 = -a \circ [e_3 + ae_2 \circ (e_3 - 2e_1) - a^2e_1] \circ W^{-1}, \tag{3.14}$$

$$W = 1 + ae_2 - (4 + 2e_1e_3) \circ a^2 + e_2 \circ (1 + 2e_1e_3) \circ a^3, \tag{3.15}$$

$$W_0 = 1 + ae_2 - (10 + 2e_1e_3) \circ a^2 + (e_2 + 14e_1e_2e_3) \circ a^3 - 4a^4, \tag{3.16}$$

and

$$\varepsilon_1 = a^2 \circ |W|^{-1} \circ |3 - 6ae_2 + 2a^2|, \quad \text{if } e_1e_3 = 1, \tag{3.17}$$

$$\varepsilon_1 = a^2 \circ |W|^{-1} \circ |2 + 5ae_2 + 2a^2|, \quad \text{if } e_1e_3 = -1. \tag{3.18}$$

Now, designate

$$G'_1 = (0, 1/2), \tag{3.19}$$

$$G'_2 = \left(0, \frac{1+\sqrt{5}}{2}\right)\setminus\{0.2234578; 0.3142734\}, \tag{3.20}$$

$$G_3' = \left(0, \frac{\sqrt{2}}{2}\right) \backslash \{0.3237679\}, \tag{3.21}$$

$$G_4' = (1/4, -1 + \sqrt{2}). \tag{3.22}$$

THEOREM 2: Assume that,

$$a \in G_1' \quad \text{if } e_1 e_3 = 1, \text{ and } e_2 = 1, \tag{3.23}$$

$$a \in G_2' \quad \text{if } e_1 e_3 = 1, \text{ and } e_2 = -1, \tag{3.24}$$

$$a \in G_3' \quad \text{if } e_1 e_3 = -1, \text{ and } e_2 = 1, \tag{3.25}$$

$$a \in G_4' \quad \text{if } e_1 e_3 = -1, \text{ and } e_2 = -1. \tag{3.26}$$

Then, the system (3.10) is the unique solution to the problem (M) in case (3.1), and H given by eq. (3.11) is the value of the infimum (1.4) in [1].

THEOREM 3: Assume that $a \in (0, 1/4)$, and $e_1 e_3 = -1$, and $e_2 = -1$. Then, the system (3.10) is an approximate solution to the problem (M). Using the criterion (K), for $a \in (0, 1/5]$ we can determine the "appropriate subsystem" (2.10) in [1], obtaining thus the CMS. For $a \in (1/5, 1/4)$ the criterion (K) fails.

The proof of the Theorems 2 and 3 is given in Appendix C.

4. DISCUSSION ON THE RESULTS

The D-algorithm discussed has the following properties.

For even input signals with two sidelobes of equal amplitude a and arbitrary sign, for $0 < |a| \leq 1/2$, it provides the exact CMS for any length of the impulse response [4, 5]. For $|a| > 1/2$ the CMS is also obtained, however in this case certain singular values of a are prohibited (see Appendix A and [5]).

For even input signals with four sidelobes of equal amplitude and arbitrary sign, assuming that the length of the impulse response equals the input signal length, ($l = 1$), the D-algorithm discussed gives an exact solution to the problem (M), provided that a takes the value from the intervals (2.5) and (2.10), (Theorem 1).

For even input signals with six sidelobes of equal amplitudes and arbitrary signs, assuming that the length of the impulse response equals the input signal length, ($l = 1$), the algorithm discussed gives exact solution to the problem (M), provided that a takes the value from the intervals given by eqs. (2.23) to (3.26), (Theorem 2).

As presented in Section 4 of [1], for the general case of input signal, the D-algorithm discussed gives an approximate solution to the problem (M). If condition (C_2), is not met then, in order to obtain the CMS we can use the criterion (K) introduced in Section 4 of [1]. The use of this criterion does not guarantee that the CMS is obtained, (see Theorem 3 for instance). However, in a few simple cases of input signal shown in Table 1, criterion (K) proved to be effective.

It is to be remembered that H and $|h|$ denote the approximation error with the D-algorithm and in the case of the CMS, respectively.

Table 1

No.	Input signal	H	$\|h\|$
1.	$a_1 = -0.1; a_2 = 0.2;$ $q = p = 2; l = 1.$	$23/656 \cong$ 0.03506097	$4/115 \cong$ 0.03478261
2.	$a_1 = 0.2; a_2 = 0.1;$ $q = p = 2; l = 1.$	$15/567 \cong$ 0.02821869	$8/307 \cong$ 0.02605863
3.	$a_1 = -0.2; a_2 = 0.1;$ $q = 2; p = 4; l = 2.$	$3.21239 \circ 10^{-3}$	$2.80336 \circ 10^{-3}$

Notably, in all cases of input signals considered, we have assumed that the sum of the sidelobe amplitude absolute values, relative to the mainlobe amplitude was less than 1. This assumption means that $\Sigma\,|a_i| < 1$, and is well known as the "input distortion $D_i < 1$", for which the Zero-Forcing-Filters represented by impulse responses $h_n^i(\hat{x}^i)$, eq. (3.3) in [1] produce the output signal with the minimum of "output distortion" [6]. The results obtained in the case of input signals with 2, 4 and 6 sidelobes of equal amplitude provide premises to expect, that in a more general case of input signal, the above assumption on the input distortion being less than 1 can be relaxed. However, in order to avoid singular points which may appear, care in checking eqs. (2.17), (2.18) in [1] and the Haar condition must be taken. Though not proved, it can also be supposed, that for even input signals with any finite number of sidelobes of equal amplitude and sign, with the input distortion $D_i < 1$, for $l = 1$, the D-algorithm discussed gives the CMS. This supposition has been confirmed in the case of input signal being the autocorrelation function of the 13 bit Barker code, ($a_i = 1\backslash 13$, $i = 1, \ldots, 6$), for which the D-algorithm solution and the CMS are identical, i.e., $H = |h| \cong 0.01821188$. Finally, it can be believed, that the D-algorithm and the criterion (K) can also be useful in the more general case of input signal, not being an even function. Obviously, for input signals with large input distortion D_i and for large values of l, the discrepancy between the D-algorithm solution and the CMS may become so large, that the criteiron (K) will fail. Application of the D-algorithm to general class of input signals as well as possible modifications of the criterion (K) are being currently examined.

APPENDIX A

The purpose of this appendix is to derive the criterion conditions, that enable us to exclude these values of a, for which, in the case of two sidelobe even input signal, the solution to the problem (M) is not obtained.

It has been proved in [5], that for arbitrary $a \neq 0$, which satisfies the following inequalities,

$$V_n(a, 1 - 2a^2) \neq 0, \quad n = 1, \ldots, k\,, \tag{A.1}$$

$$\sum_{i=1}^{k} (-1)^{i+1} \circ e_{i+1} \circ a^{-i+1} \circ V_{i-1}(a, 1) \neq (2K)^{-1}\,, \tag{A.2}$$

(see ineqs. (2.10) and (2.21) in [5]), the minimax problem M_h, which is equivalent to the problem (M) formulated in Section 1 of [1], has the unique solution. It was also shown, that for $0<|a|\leqslant 1/2$, (A.1) and (A.2) are always satisfied, which implies the unconditional existence of the unique solution. This particular case has been dealt with in detail in [4].

Now, consider another form of ineq. (A.2) for $|a|>1/2$. Using the theory of difference equations, (e.g. [7]), we obtain the solution to eq. (2.9) in [5], in the following form,

$$d_n = C_1 \circ \cos(n\alpha) + C_2 \circ \sin(n\alpha), \quad n = 1, \ldots, k+1, \tag{A.3}$$

where

$$\cos\alpha = -(2a)^{-1}, \quad \sin\alpha = -C\cdot(2a)^{-1}, \quad C = (4a^2-1)^{1/2}, \tag{A.4}$$

and C_1, C_2 are arbitrary constants independent of n.

Taking into account the inequality (2.8) in [5] and putting $d_1 = 1$ we obtain,

$$d_n = -2a \circ \cos(n\alpha), \quad n = 1, \ldots, k+1. \tag{A.5}$$

For $|a|>1/2$, the solution of the recurrence equation (2.2) in [5], is given as,

$$V_n(a, b) = (-a)^n \circ [D_1 \cos(n\alpha) + D_2 \sin(n\alpha)], \quad n = 1, 2, \ldots \tag{A.6}$$

where D_1 and D_2 are arbitrary constants. Taking into account eqs. (2.3) in [5], we have the following eqs.

$$V_n(a, b) = (-a)^n \circ [\cos(n\alpha) + C^{-1}(2b-1)\sin(n\alpha)], \quad n = 0, 1, \ldots. \tag{A.7}$$

Hence, it follows that,

$$V_n(a, 1-2a^2) = 2(-a)^{-n+1} \circ \cos(n+1)\alpha, \quad n = 0, 1, \ldots \tag{A.8}$$

Eqs. (A.5) and (A.8) imply that,

$$d_n = (-a)^{-n+1} \circ V_{n-1}(a, 1-2a^2), \quad n = 1, 2, \ldots, k+1. \tag{A.9}$$

Hence, taking into account eqs. (2.15) and (2.17) in [5] we find that ineq. (A.2) takes the following form,

$$a^k + \sum_{i=1}^{k} \hat{e}_i \circ a^{k-i} \circ [V_i(a, 1-2a^2) + 2a^2 \circ V_{i-1}(a, 1)] \neq 0, \tag{A.10}$$

where

$$\hat{e}_i = (\operatorname{sgn} a)^i \circ \operatorname{sgn} V_i(a, 1-2a^2), \quad i = 1, \ldots, k. \tag{A.11}$$

The inequalities (A.1) and (A.10) formulate the necessary and sufficient condition for the existence of the unique solution to the minimax problem M_h in [5] for $|a|>1/2$. To solve these inequalities we use formulae (2.2) and (2.3) in [5] and derive $V_i(a, b)$ for $b = 1$ and for $b = 1 - 2a^2$ in the form of polynomials of the variable a. Now, let Z_k be a system of all real numbers, $a \neq 0$, for which the problem M_h has the unique solution. Then, we have,

$$Z_1 = R \backslash \{0, \pm 1/\sqrt{2}, \pm 1\} ,$$

$$Z_2 = R \backslash \{0, \pm /\sqrt{3}, \pm 1/\sqrt{2}, \pm 1\} ,$$

where R is the system of all real numbers.

Derivation of Z_k for larger values of k requires tedious calculations. Therefore, it was found to be more convenient to check ineqs. (A.1) and (A.10) for the assumed value of k, finding thus, whether the particular value of a belongs to Z_k. To illustrate this way consider for instance $k = 4$ and $a = 2$. Then, by eqs. (2.2) and (2.3) in [5] we have,

$$V_0(2, -7) = 1,\ V_1(2, -7) = -7,\ V_2(2, -7) = -11,\ V_3(2, -7) = 17$$
$$V_4(2, -7) = 61,\ V_0(2, 1) = 1,\ V_1(2, 1) = 1,\ V_2(2, 1) = -3 ,$$
$$V_3(2, 1) = -7 .$$

Hence, by eqs. (A.11) it follows that,

$$\hat{e}_1 = -1, \quad \hat{e}_2 = -1, \quad \hat{e}_3 = 1, \quad \hat{e}_4 = 1 ,$$

and we easily find that the inequalities (A.1) and (A.10) are satisfied, which means, that $2 \in Z_4$.

Finally, it should be noted, that the unconditional existence of the solution to M_h for $0 < |a| \leq 1/2$ means, that for any value of k

$$[-1/2, 0) \cup (0, 1/2] \subset Z_k$$

APPENDIX B

The purpose of this appendix is to prove Theorem 1, formulated in Section 2.

Proof of Theorem 1: From eqs. (1.1) to (1.3) in [1] and eq. (3.1) it follows that,

$$\left.\begin{aligned} g_0(x) &= 1 + 2a \circ (e_1 x_1 + e_2 x_2) , \\ g_1(x) &= ae_1 + (ae_2 + 1) \circ x_1 + ae_1 x_2 , \\ g_2(x) &= ae_2 + ae_1 x_1 + x_2 , \\ g_3(x) &= a \circ (e_2 x_1 + e_1 x_2) , \\ g_4(x) &= ae_2 x_2 . \end{aligned}\right\} \qquad \text{(B.1)}$$

With reference to the D-algorithm we solve the following system of equations, (3.5) in [1]:

$$g_1(x) = 0,\ g_2(x) = 0\,. \tag{B.2}$$

Definitions (2.2) and (2.3) together with assumptions (2.4) and (2.5) secure the existence of the unique solution $x' = (x_1', x_2')$ to this system,

$$x_1' = \frac{ae_1 \circ (ae_2 - 1)}{1 + ae_2 - a^2}\,, \quad x_1' = \frac{-ae_2}{1 + ae_2 - a^2}\,. \tag{B.3}$$

Hence, and from eq. (2.9) it follows that,

$$\left.\begin{aligned} g_0(x') &= B \circ (1 + ae_2 - a^2)^{-1}\,, \\ g_3(x') &= -\frac{a^2 e_1 \circ (2e_2 - a)}{1 + ae_2 - a^2}\,, \\ g_4(x') &= -\frac{a^2}{1 + ae_2 - a^2}\,. \end{aligned}\right\} \tag{B.4}$$

From assumptions (2.5) and (2.10) it follows that,

$$|g_4(x')| < |g_3(x')| = \varepsilon_1\,. \tag{B.5}$$

From eqs. (3.16) and (3.17) in [1] it follows that,

$$H = \frac{a\varepsilon_1}{\varepsilon_1 + a \circ |g_0(x')|}\,, \ \hat{x}_n = \frac{ax_n'}{a + \varepsilon_1 \circ \operatorname{sgn} g_0(x')}\,, \quad n = 1, 2\,, \tag{B.6}$$

which consequently gives formulae (2.6) to (2.8).

Now, using the variables linearisation method, (Section 2 of [1]), we have,

$$r_n(t) = g_n(x) \circ [g_0(x)]^{-1}, \quad n = 1, 2, 3, 4\,, \tag{B.7}$$

where

$$\left.\begin{aligned} r_1(t) &= ae_1 + (1 - 2a^2 + ae_2) \circ t_1 + ae_1 \circ (1 - 2ae_2) \circ t_2\,, \\ r_2(t) &= ae_2 + ae_1 \circ (1 - 2ae_2) \circ t_1 + (1 - 2a^2) \circ t_2\,, \\ r_3(t) &= a \circ (e_2 t_1 + e_1 t_2),\ r_4(t) = ae_2 t_2\,. \end{aligned}\right\} \tag{B.8}$$

With reference to eqs. (3.6) and (3.11) in [1] we have,

$$\hat{g}_n^1(x) = z_1 \circ s(n) + g_n(x), \quad z_1 = \frac{\varepsilon_1}{a} \circ \operatorname{sgn} g_0(x')\,. \tag{B.9}$$

Thence, by relation (B.5) it follows that,

$$|\hat{g}_1^1(x')| = |\hat{g}_2^1(x')| = |\hat{g}_3^1(x')| > |\hat{g}_4^1(x')|\,. \tag{B.10}$$

Having applied the criterion (K) we choose $N = \{1, 2, 3\}$ as the appropriate subsystem of $N_0 = \{1, 2, 3, 4\}$. Consequently, using eq. (2.12) in [1] we have,

$$\begin{aligned} &(1 - 2a^2 + ae_2) \circ \lambda_1 + ae_1 \circ (1 - 2ae_2) \circ \lambda_2 + ae_2 \lambda_3 = 0 , \\ &ae_1 \circ (1 - 2ae_2) \circ \lambda_1 + ae_1 \circ (1 - 2ae_2) \circ \lambda_2 + ae_2 \lambda_3 = 0 . \end{aligned} \tag{B.11}$$

Assuming $\lambda_3 = -1$, by eq. (2.9) we obtain

$$\lambda_1 = a \circ (e_2 - a) \circ B^{-1}, \quad \lambda_2 = ae_1 B^{-1} . \tag{B.12}$$

From assumptions (2.4) and (2.5) it follows that,

$$\lambda_n \neq 0, \quad \text{for } n = 1, 2, 3 , \tag{B.13}$$

which is equivalent to the Haar condition for the vectors:

$$\left.\begin{aligned} &A_1 = [1 - 2a^2 + ae_2, ae_1 \circ (1 - 2ae_2)] , \\ &A_2 = [ae_1 \circ (1 - 2ae_2), 1 - 2a^2] , \\ &A_3 = [ae_2, ae_1] . \end{aligned}\right\} \tag{B.14}$$

Thus it follows that the problem $(\hat{M})$ for the assumptions (2.1) has the unique solution. From eq. (2.12) in [1] and eq. (B.12) we obtain,

$$h = \frac{a^2 e_1 \circ (2e_2 - a) \circ \operatorname{sgn} B}{a \circ (|e_2 - a| + 1) + |B|} . \tag{B.15}$$

Hence, and from eq. (B.6) it follows that $|h| = H$, for assumptions (2.5) and (2.10), which completes the proof of Theorem 1.

Finally, it is to mention that the conditions (2.3) to (2.5) result from the following inequalities,

$$a > 0, \quad 1 - ae_2 - a^2 \neq 0, \quad \text{(B.5)}, \quad B \neq 0 .$$

One can easily find that these inequalities imply ineq. (B.13).

APPENDIX C

The purpose of this appendix is to prove Theorem 2 and Theorem 3 formulated in Section 3.

Proof of Theorem 2: From eqs. (1.1) to (1.3) in [1] and eq. (3.1) it follows that,

$$\left.\begin{aligned} g_0(x) &= 1 + 2a\circ(e_1x_1 + e_2x_2 + e_3x_3)\,,\\ g_1(x) &= ae_1 + (ae_2+1)\circ x_1 + a\circ(e_1+e_3)\circ x_2 + ae_2x_3\,,\\ g_2(x) &= ae_2 + a\circ(e_1+e_3)\circ x_1 + x_2 + ae_1x_3\,,\\ g_3(x) &= ae_3 + ae_2x_1 + ae_1x_2 + x_3\,,\\ g_4(x) &= a\circ(e_3x_1 + e_2x_2 + e_1x_3)\,,\\ g_5(x) &= a\circ(e_3x_2 + e_2x_3)\,,\\ g_6(x) &= ae_3x_3\,. \end{aligned}\right\} \tag{C.1}$$

Using the D-algorithm, we solve the following system of equations, (3.5) in [1]:

$$g_n(x) = 0, \quad n = 1, 2, 3\,. \tag{C.2}$$

Assumptions (3.6) to (3.9) together with definitions (3.2) to (3.5) imply the existence of the unique solution to (C.2), $x' = (x_1', x_2', x_3')$ given by formulae (3.12) to (3.14). Hence, we have,

$$\left.\begin{aligned} g_4(x') &= -a^2W^{-1}\circ[1 + 2e_1e_3 - a\circ(4e_2 + 2e_1e_2e_3) + a^2\circ(e_1e_3+1)]\,,\\ g_5(x') &= -a^2W^{-1}\circ[2e_2e_3 + a\circ(e_3 - 4e_1) + a^2\circ(e_2e_3 - e_2e_1)]\,,\\ g_6(x') &= -a^2W^{-1}\circ[1 + ae_2\circ(1-2e_1e_3) - a^2e_1e_3]\,, g_0(x') = W_0\circ W^{-1}\,. \end{aligned}\right\} \tag{C.3}$$

where W and W_0 are given by eqs. (3.15) and (3.16), respectively. Relations (3.2) to (3.9) also imply that,

$$\varepsilon_1 = \max_{4\le n\le 6} |g_n(x')| = |g_4(x')|, \quad \text{for } e_1e_3 = 1\,, \tag{C.4}$$

$$\varepsilon_1 = \max_{4\le n\le 6} |g_n(x')| = |g_5(x')|, \quad \text{for } e_1e_3 = -1\,. \tag{C.5}$$

By eqs. (3.16) and (3.17) in [1] and the above formulae one obtains eqs. (3.10), (3.11), (3.17) and (3.18).

Now, using the variable linearisation method, (Section 2 of [1]) and designating $r_n = g_ng_0^{-1}$, $n = 1, \ldots, 6$, we have,

$$\begin{aligned} r_1(t) &= ae_1 + (ae_2 + 1 - 2a^2)\circ t_1 + a\circ(e_1 + e_3 - 2ae_1e_2)\circ t_2 + a\circ(e_2 - 2ae_1e_3)\circ t_3\,,\\ r_2(t) &= ae_2 + a\circ(e_1 + e_3 - 2ae_1e_2)\circ t_1 + (1-2a^2)\circ t_2 + a\circ(e_1 - 2ae_2e_3)\circ t_3\,,\\ r_3(t) &= ae_3 + a\circ(e_2 - 2ae_1e_3)\circ t_1 + a\circ(e_1 - 2ae_2e_3)\circ t_2 + (1-2a^2)\circ t_3\,,\\ r_4(t) &= a\circ(e_3t_1 + e_2t_2 + e_1t_3)\,,\\ r_5(t) &= a\circ(e_3t_2 + e_2t_3)\,,\\ r_6(t) &= ae_3t_3\,. \end{aligned} \tag{C.6}$$

In order to use criterion (K) let,

$$\hat{g}_n^1(x) = z_1 \circ s(n) + g_n(x), \quad z_1 = \frac{\varepsilon_1}{a} \circ \operatorname{sgn} g_0(x') \,. \tag{C.7}$$

First, consider the case $e_1 e_3 = 1$.
Then, from eq. (C.4) it follows that,

$$|\hat{g}_1^1(x')| = |\hat{g}_2^1(x')| = |\hat{g}_3^1(x')| = |\hat{g}_4^1(x')| > |\hat{g}_n^1(x')| \,, \quad \text{for } n = 5, 6 \,. \tag{C.8}$$

Having applied the criterion (K) we choose $N = \{1, 2, 3, 4\}$ as the appropriate subsystem of $N_0 = \{1, \ldots, 6\}$. Consequently, using eq. (2.12) in [1] we have,

$$\left.\begin{aligned} &(e_2 a + 1 - 2a^2) \circ \lambda_1 + 2ae_1 \circ (1 - ae_2) \circ \lambda_2 + a \circ (e_2 - 2a) \circ \lambda_3 + ae_1 \lambda_4 = 0 \,, \\ &2ae_1 \circ (1 - ae_2) \circ \lambda_1 + (1 - 2a^2) \circ \lambda_2 + ae_1 \circ (1 - 2ae_2) \circ \lambda_3 + ae_2 \lambda_4 = 0 \,, \\ &a \circ (e_2 - 2a) \circ \lambda_1 + ae_1(1 - 2ae_2) \circ \lambda_2 + (1 - 2a^2) \circ \lambda_3 + ae_1 \lambda_4 = 0 \,. \end{aligned}\right\} \tag{C.9}$$

Thence, puting $\lambda_1 = -e_1$ and using eq. (3.16) we have,

$$\lambda_2 = \frac{e_2 + a}{1 - 2ae_2}, \quad \lambda_3 = \frac{e_1 \circ (-1 + e_2 a + a^2)}{(1 - ae_2) \circ (1 - 2ae_2)} \,, \tag{C.10}$$
$$\lambda_4 = W_0 \circ [a(1 - ae_2) \circ (1 - 2ae_2)]^{-1} \,.$$

Let $A_i \in R^3$, $(i = 1, \ldots, 6)$, be a vector, whose coordinates equal the coefficients of the variables t_1, t_2, t_3 of $r_i(t)$. From eqs. (3.23) and (3.24) it follows that,

$$\lambda_n \neq 0, \quad n = 1, 2, 3, 4 \,, \tag{C.11}$$

which is equivalent to the Haar condition for the vectors A_i, A_2, A_3, A_4. Thence, the minimax problem ($\hat{M}$), for the case (3.1) has the unique solution. Formulae (C.10) and (2.13) in [1] imply that,

$$h = -\frac{a^2 \circ (3 - 6ae_2 + 2a^2) \circ \operatorname{sgn}(1 - 3ae_2 + 2a^2)}{aB + |W_0|} \,, \tag{C.12}$$

where $B = |1 - 2ae_2 + 2a^2| + |a^2 + ae_2 - 1| + (a - e_2)^2$.

Taking into account assumptions (3.23), (3.24) and relations (3.11), (3.17) we obtain $|h| = H$, which, for the case $e_1 e_3 = 1$ completes the proof.

Now, consider the case $e_1 e_3 = -1$.

Then, from eqs. (C.5) and (C.7) it follows that,

$$|\hat{g}_1^1(x')| = |\hat{g}_2^1(x')| = |\hat{g}_3^1(x')| = |\hat{g}_5^1(x')| > |\hat{g}_n^1(x')| \,, \quad \text{for } n = 4, 6 \,. \tag{C.12}$$

Having applied the criterion (K) we choose $N = \{1, 2, 3, 5\}$ as the appropriate subsystem of $N_0 = \{1, \ldots, 6\}$. Consequently, using eq. (2.12) in [1] we have,

$$\left.\begin{aligned} &(e_2a+1-2a^2)\circ\lambda_1-2a^2e_1e_2\lambda_2+a\circ(e_2+2a)\circ\lambda_3=0\,,\\ &-2a^2e_1e_2\lambda_1+(1-2a^2)\circ\lambda_2+ae_1\circ(1+2ae_2)\circ\lambda_3-ae_1\lambda_5=0\,,\\ &a\circ(e_2+2a)\circ\lambda_1+ae_1(1+2ae_2)\circ\lambda_2+(1-2a^2)\circ\lambda_3+ae_2\lambda_5=0\,.\end{aligned}\right\} \qquad (C.13)$$

Thence, putting $\lambda_1=-e_1$ and using eq. (3.16) we have,

$$\left.\begin{aligned} &\lambda_2=(-1-2ae_2+2a^2+4a^3e_2)\circ[a\circ(1+5ae_2+4a^2)]^{-1}\,,\\ &\lambda_3=e_1\circ(e_2+2a-a^2e_2)\circ[a\circ(1+5ae_2+4a^2)]^{-1}\,,\\ &\lambda_5=-e_1W_0\circ[a^2\circ(1+5ae_2+4a^2)]^{-1}\,.\end{aligned}\right\} \qquad (C.14)$$

From assumptions (3.25) and (3.26) it follows that,

$$\lambda_n\neq 0,\quad n=1,2,3,5\,, \qquad (C.15)$$

which is equivalent to the Haar condition for the vectors A_1, A_2, A_3, A_5. Thence, the minimax problem ($\hat{M}$), for the case (3.1) has the unique solution. Formulae (C.14) and (2.13) in [1] imply that,

$$h=-\frac{a^2e_2\circ(2+5ae_2+2a^2)\circ\operatorname{sgn}(1+5ae_2+4a^2)}{aB+|W_0|}\,, \qquad (C.16)$$

where $B=a\circ|1+5ae_2+4a^2|+|-a^2+2ae_2+1|+|1+2ae_2-2a^2-4a^3e_2|$.

Taking into account assumptions (3.25), (3.26) and relations (3.11), (3.18) we obtain $|h|=H$, which, for the case $e_1e_3=-1$ completes the proof.

Proof of Theorem 3: Assumptions of this theorem show, that all considerations of Theorem 2 for the case $e_1e_3=-1$, are also valid here. In addition we have the following equalities

$$\operatorname{sgn}\lambda_1=-e_1,\quad \operatorname{sgn}\lambda_2=1,\quad \operatorname{sgn}\lambda_3=-e_1,\quad \operatorname{sgn}\lambda_4=-e_1\,, \qquad (C.17)$$

and

$$h=\frac{a^2\circ(2-5a+2a^2)}{1+a-11a^2+5a^3+4a^4},\quad H=\frac{a^2\circ(2-5a+2a^2)}{1+a-13a^2+15a^3-4a^4}\,. \qquad (C.18)$$

One can easily find that $0<h<H$ for $a\in(0;1/4)$, which completes the proof of the first part of Theorem 3.

In order to prove the second part of this theorem, by criterion (K) we take the following system of equations,

$$r_n(t)=h\circ\operatorname{sgn}\lambda_n,\quad n=2,3,5\,, \qquad (C.19)$$

which implies,

$$\left.\begin{aligned}&2a^2e_1t_1+(1-2a^2)\circ t_2+ae_1\circ(1-2a)\circ t_3=-h+a\,,\\&a\circ(-1+2a)\circ t_1+ae_1\circ(1-2a)\circ t_2+(1-2a^2)\circ t_3=-e_1h+ae_1\,,\\&at_2+ae_1t_3=h\,.\end{aligned}\right\}\quad (C.20)$$

The solution to eqs. (C.20) is the following system of numbers,

$$\left.\begin{aligned}\hat{t}_1&=\frac{2e_1}{4a-1}+\frac{he_1\circ(4a^2-3a-1)}{a^2\circ(4a-1)}\,,\\\hat{t}_2&=\frac{4a^2-2a}{(1-a)\circ(4a-1)}+\frac{h\circ(8a^2-2a-1)}{a\circ(1-4a)}\,,\\\hat{t}_3&=\frac{ae_1}{(1-a)\circ(4a-1)}+\frac{he_1}{a\circ(1-4a)}\,.\end{aligned}\right\}\quad (C.21)$$

Hence,

$$r_4(\hat{t})=a\circ(-e_1\hat{t}_1-\hat{t}_2+e_1\hat{t}_3)\,,\quad (C.22)$$

which, by eqs. (C.18) and (C.21) gives,

$$r_4(\hat{t})=\frac{a^2\circ(1+2a-8a^2)}{1+a-11a^2+5a^3+4a^4}>0,\quad \text{for } a\in(0,1/4)\,.\quad (C.23)$$

Now, using eq. (C.18) we find the following relations

$$\left.\begin{aligned}&|r_4(\hat{t})|<|h|,\quad \text{for } 0<a<1/5\,,\\&|r_4(\hat{t})|=|h|,\quad \text{for } a=1/5\,,\\&|r_4(\hat{t})|>|h|,\quad \text{for } 1/5<a<1/4\,.\end{aligned}\right\}\quad (C.24)$$

We also have,

$$|r_6(\hat{t})|<|h|,\quad \text{for } a\in(0,1/4)\,.\quad (C.25)$$

Finally, by ineq. (2.17) in [1], and eqs. (C.18) and (C.21) we have,

$$1-2a\circ(e_1\hat{t}_1-\hat{t}_2-e_1\hat{t}_3)\neq 0,\quad \text{for } a\in(0,1/4)\,,\quad (C.26)$$

which completes the proof of the Theorem 3.

This proof also implies, that for $a\in(0,1/5)$, the unique solution to the problem (M), with assumptions (3.1), in the case of $e_1e_3=1$, $e_2=-1$ is given as

$$\hat{x}_i=\hat{t}_i\circ[1-2a\circ(e_1\hat{t}_1-\hat{t}_2-e_1\hat{t}_3)]^{-1}\,,\quad \text{for } i=1,2,3\,.\quad (C.27)$$

Finally, it should be noted, that the relations (3.2), (3.3) and the assumptions (3.6) and (3.7) were derived from the following inequalities:

$$a > 0, \quad (C.8), \quad W \neq 0, \quad W_0 \neq 0 .$$

Relations (3.4), (3.5) and the assumptions (3.8) and (3.9) were derived from the inequalities:

$$a > 0, \quad (C.12), \quad W \neq 0, \quad W_0 \neq 0 .$$

Assumptions (3.6) to (3.9) were sufficient for using the D-algorithm. However, verification of this algorithm also required checking ineq. (C.11) for $e_1 e_3 = 1$ and ineq. (C.15) for $e_1 e_3 = -1$, respectively.

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