Theorem 4

The 2-DFMM is c-reconstructible *iff* the pair $(C, A_1 r + A_2 s)$ is reconstructible in 1-D sense for all $r, s \subset C$.

Proof:

a) Sufficiency: Assuming $u = 0$ and using backward shift operators w and z, where

$$z^j f(k,t) = f(k, t-j)$$

the model equation (1) can be presented as follows:

$$A(w,z)x(k,t) = \begin{bmatrix} 0 \\ y(k,t) \end{bmatrix}$$

where

$$A(w,z) = \begin{bmatrix} I - A_1 z - A_2 w \\ C \end{bmatrix}.$$

If the theorem's condition is fulfilled rank $A(w,z) = n$ for all $w, z \subset C$, [6]. Therefore, there exists a multinomial matrix $H(w,z) = [H_1(w,z), H_2(w,z)]$ such that $H(w,z)A(w,z) = I$, [12]. Hence

$$H_2(w,z)y(k,t) = H(w,z)A(w,z)x(k,t) = x(k,t).$$

b) Necessity: It is easy to note that exist w_f, z_f and x_f such that $x_f + x_f^* \neq 0$ and $A(w_f, z_f)x_f = 0$ if theorem's condition is not fulfilled. Then, for $X_2(0)$ such that

$$x(i,0) = w_f^{k-i} x_f, \qquad i = 1, \cdots, k$$

and

$$x(0,j) = z_f^{t-j} x_f, \qquad j = 1, \cdots, t.$$

One gets $x(i,j) = w_f^{k-i} z_f^{t-j} x_f$ for $i = 1, \cdots, k$ and $j = 1, \cdots, t$. Since $Cx_f = 0$ the nonzero real l-state $x(k,t) = x_f + x_f^*$ is indistinguishable from zero for $i \leqslant k$ and $j \leqslant t$ similarly as in previous theorems.

Remarks

1) It is easy to note that 2-DFMM is c-reconstructible *iff* it is g_1-reconstructible and the pairs (C, A_1) and (C, A_2) are reconstructible.

2) It follows from the theorem that well-known tests for checking 1-D systems reconstructibility can be used for verification of 2-DFMM c-reconstructibility, e.g., those given in [1].

Note, that similarly to observability, so-called g-reconstructibility is not a global property of a system, i.e., for instance g_1-reconstructibility does not imply g_2-reconstructibility. We illustrate it with an example.

Example 2

Consider 2-DFMM with $A_1 = \begin{bmatrix} 1,0 \\ 0,0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0,0 \\ 1,0 \end{bmatrix}$, and $C = [0,1]$. It is easy to check that the system is c-observable. Nevertheless, this system is c-unreconstructible: appropriate matrix $A(w,z)$ from proof of Theorem 4 has rank 1 for $w = 1$ and $z = 0$. Thus in the contradiction to g_1-observability, c-observability does not imply c-reconstructibility of 2-DFMM, and vice vérsa, c-unreconstructibility does not imply its c-unobservability.

Moreover, since c-observability implies g_1-reconstructibility, Remarks 2 and 1 to Theorems 1 and 3, and the system is g_2-reconstructible *iff* it is c-reconstructible, it implies that g_1-reconstructibility is not g-property of the system.

IV. Concluding Remarks

The concepts of g_i- and c-observability and reconstructibility for 2-DFMM are presented. It is shown that c-observability and reconstructibility are global properties of the system—they imply respectively its g_i-observability and reconstructibility. It is also obvious that g_i-unobservability and unreconstructibility imply c-unobservability and unreconstructibility of 2-DFMM. Moreover, since 2-DFMM is causal, g_1-unobservability and unreconstructibility are also global properties of a system. Relations between the properties can be briefly presented as follows.

$$\left.\begin{array}{l} c\text{-observability} \Rightarrow g_1\text{-observability} \\ c\text{-reconstructibility} \end{array}\right\} \Rightarrow g_1\text{-reconstructibility}$$

g_1-unreconstructibility

$$\Rightarrow \left\{\begin{array}{l} g_1\text{-unobservability} \Rightarrow c\text{-unobservability} \\ c\text{-unreconstructibility}. \end{array}\right.$$

Let us also note that presented results can be extended simply on N-DFMM because relevant theorems from [12] deal with N-D multinomial matrices.

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Minimax Sidelobe Reduction Filtering for Huffman Sequence Autocorrelation Function Type Signals

ANDRZEJ DYKA AND HENRY UGOSKI

Abstract—**This paper presents a solution for the impulse response of an FIR sidelobe reduction filter (SRF), intended for signals consisting of a mainlobe and two symmetrically spaced sidelobes, i.e., for signals of the Huffman sequence autocorrelation function type, which maximizes the output mainlobe level subject to a constraint on the maximum sidelobe level. It is shown, that for those input signals whose sidelobe level does not exceed half of the mainlobe level, the filter required can be represented uniquely as a parallel combination of zero forcing filters. The tradeoff between the reduction of sidelobe level and the degradation of signal-to-noise ratio is examined.**

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The authors are with the Institute of Telecommunications, Politechnika Gdansk, Gdansk, Poland.

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I. Introduction

The Huffman sequences (H-S) belong to the class of wideband signals which can be used for matched filter pulse compression in echo-ranging systems, e.g., [3]. The output signal of a matched filter is the autocorrelation function (ACF) of the input signal. The ACF of a H-S is a sequence $\{s(n)\}$, which consists of three nonzero elements, with the following amplitudes:

$$\left.\begin{aligned}&\{s(n)\} = \delta(n) + \frac{\alpha}{2}\circ[\delta(n-1)+\delta(n+1)]\\ &\text{where}\\ &\delta(n) \triangleq \begin{Bmatrix}0, & \text{for } n\neq 0\\ 1, & \text{for } n=0\end{Bmatrix}, \qquad \alpha = \mp\frac{1}{E}\end{aligned}\right\} \quad (1)$$

E denotes the H-S energy.

The element of amplitude 1 appearing at $n=0$, is usually referred to as the central peak or the mainlobe, while the other two taking values of $\alpha/2$ are called the sidelobes.

The occurrence of sidelobes in any system is, as a rule, an undesirable phenomenon. For instance, in echo-ranging systems the output signal sidelobes lead to ambiguous detection. The only way to reduce the degradation of system performance caused by sidelobes is to use special filtering, which decreases their amplitude relative to the amplitude of the mainlobe. The processor which implements such filtering is called the sidelobe reduction filter and will be hereafter referred to as the SRF. Notably, the problem of sidelobe reduction filtering is basically equivalent to the problem of linear deconvolution filtering. In an ideal deconvolution filter case one seeks a solution for the impulse response of a filter, which produces the output signal in the form of a single sidelobe-free mainlobe. However, as the appearance of output signal sidelobes is, in the case of an FIR filter, inevitable, the realistic approach to the linear deconvolution filtering problem consists in using an SRF defined above. As the SRF's usually follow the matched filter, their overall performance, in the sense of signal-to-noise ratio, is inferior to the performance of the matched filter alone. Therefore, the problem of determining an SRF structure is usually solved by using complex optimization procedures which, in general, compromise between maximum reduction of sidelobe level and minimum degradation of signal-to-noise ratio, e.g., [1], [5]. Filters obtained in this way are usually called "optimum filters." It should be noted that the problem of sidelobe reduction filtering is also closely related to channel equalization in digital communication. Basically, equalization consists in adaptive filtering, which reduces the sidelobes of the time varying impulse response of a transmission channel, e.g., [7], [9].

In this paper a solution for the impulse response of an FIR type SRF for signals such as the H-S ACF's, which maximizes the mainlobe amplitude, subject to a constraint on the maximum sidelobe, is presented. The paper organization is as follows. Section II formulates the filtering problem, Section III gives the general solution, Section IV presents, in brief, noise considerations and Section V gives experimental results. Appendix A proves that the SRF impulse response must be an even function. Appendix B gives the derivation of the SRF output PSLR.

II. Formulation of Filtering Problem

Prior to the filtering problem formulation we recall the peak, sidelobe ratio (PSLR) relative to the mainlobe amplitude, defined as [8]

$$\mathrm{PSLR} = 10\log\left(\frac{S_{\max}^2}{M^2}\right), \quad [\mathrm{dB}] \qquad (2)$$

where $S_{\max}$ denotes the amplitude of the maximum sidelobe and M denotes the amplitude of the mainlobe.

The objective here is to find a solution for an impulse response of the FIR type SRF, which maximizes the mainlobe level, subject to a constraint on the maximum sidelobe, i.e., the filter which minimizes the output signal PSLR. In other words, we seek a sequence $\{h(n)\}$, which after convolution with the input signal $\{s(n)\}$ would produce an output signal $\{g(n)\}$ with the minimum PSLR. Thus one can write

$$\{g(n)\} = \{s(n)\} * \{h(n)\} \qquad (3)$$

where the asterisk denotes convolution.

One can prove that if the minimum PSLR is to be attained, then the fact that the input signal is an even sequence implies that the impulse response $\{h(n)\}$ should be an even sequence as well, (for proof see Appendix A). Consequently, it implies that for the output signal $\{g(n)\}$ is an even sequence too. Now, let us denote the impulse response $\{h(n)\}$ required as follows:

$$\{h(n)\} = \delta(n) + \sum_{\substack{i=-k\\ i\neq 0}}^{k} h(i)\circ\delta(n-i), \qquad k=1,2,\cdots \qquad (4)$$

where $h(i)$ denotes the weights of the SRF impulse response required. Substitution of (1) and (4) into (3) yields

$$\{g(n)\} = g(0)\circ\delta(n) + \sum_{i=-(k+1)}^{k+1} g(i)\circ\delta(n-i) \qquad (5)$$

where $g(0) = 1+\alpha\circ h(1)$-denotes the mainlobe amplitude in the SRF output signal, and $g(i)$, $i\neq 0$ denotes the sidelobe amplitudes in the SRF output signal.

Consequently, one can formulate our filtering problem as the determination of the $\{h(n)\}$ subject to the following minimax criterion:

$$\left[\frac{\max(|g(i)|,\ i\neq 0)}{|g(0)|}\right] \qquad (6)$$
$$\{h(n)\}\in H$$

where H is the set of all functions $\{h(n)\}$, which take the value 0, for $n=\pm(k+1), \pm(k+2),\cdots,$.

III. Derivation of Impulse Response

The problem of sidelobe reduction filtering formulated in Section II can be solved using the following theorem.

Theorem 1: If $|\alpha|\leqslant 1$ and the weights $h(n)$ of the $\{h(n)\}$ impulse response are the solutions of the following system of equations:

$$|g(1)| = |g(k+1)|, |g(2)| = |g(k+1)|, \cdots, |g(k)| = |g(k+1)| \qquad (7)$$

then only one of these solutions satisfies criterion (6).

Remarkably, *Theorem 1* states that all sidelobes in the SRF output signal, which satisfies (6), i.e., with the minimum PSLR, have equal amplitudes. The proof for this theorem as well as the general solution for (7) is given in [10]. It is also shown there, that for those input signals $\{s(n)\}$ which satisfy $|\alpha|>1$, the SRF subject to criterion (6) suffers from a poor output signal-to-noise ratio. Therefore, we will focus our attention on the class of input signals $\{s(n)\}$ which satisfy the following condition:

$$|\alpha|\leqslant 1 \qquad (8)$$

as, in addition the following unique solution to (7) for this class exists.

If the input signal $\{s(n)\}$ satisfies (8), then the solution for the SRF impulse response $\{h(n)\}$ can be written as

$$h(0)=1, \quad h(n)=p_n/p_0, \qquad \text{for } n=1,2,\cdots,k \tag{9}$$

where p_n can be computed from the following recurrence formula [10]:

$$p_k=2/\alpha, \quad p_{k-1}=-2p_k/\alpha-2e_k/\alpha,\cdots$$
$$p_n=-2p_{n+1}/\alpha-p_{n+2}-2e_{n+1}/\alpha, \qquad \text{for } n=k-2,\cdots,1,0. \tag{10}$$

$$e_n=[\operatorname{sgn}(-\alpha)]^{k-n}, \qquad \text{where } \operatorname{sgn}(x)=\begin{cases}1, & x\geqslant 0\\ -1, & x<0.\end{cases} \tag{11}$$

The sign of the consecutive samples $h(n)$ can be found as

$$\operatorname{sgn} h(n)=[\operatorname{sgn}(-\alpha)]^n. \tag{12}$$

We also have the following:

$$|h(1)|>\cdots>|h(k)|. \tag{13}$$

The properties of the SRF output signal $\{g(n)\}$ can be shown as

$$g(0)>0, \quad g(n)=-e_n\circ g(k+1),$$
$$\operatorname{sgn}[g(n)]=[\operatorname{sgn}(-\alpha)]^{n+1}, \tag{14}$$

for $n=1,\cdots,k+1$.

Although the above formulas, namely (9)–(14), allow us to determine the impulse response $\{h(n)\}$ for the SRF required, it is worth mentioning that the latter can be uniquely shown[1] as the parallel combination of finite length zero forcing filters ZFF's, [9].

As can be seen in Fig. 1 the impulse response $\{h(n)\}$ can be represented as the sum of properly delayed component impulse responses $\{h_i(n)\}$, $i=0,1,2,\cdots,k$ which represent finite length ZFF's, with the corresponding lengths of $2i$. In this case the filter length is understood as the number of unit distances between samples in its impulse response. By *Theorem 1*, all component impulse responses $\{h_i(n)\}$ should be weighted in such a way that the absolute values of all ZFF output sidelobe amplitudes are equal. Then the absolute values of the SRF output sidelobe amplitudes are equal too, while the mainlobe amplitude equals the sum of all ZFF's mainlobe amplitudes. It is obvious that the greater the reduction of sidelobe level required, the larger the number, k, of ZFF's which must be taken into account. The number k also gives the value of the delay between the output mainlobe $g(0)$ of the $\{g(n)\}$ and the input mainlobe $s(0)$ of the $\{s(n)\}$.Moreover, the length of the SRF equals k multiplied by the input signal length. Hence, an SRF specified by the number k will be hereafter called the kth order SRF.

In order to determine $\{h(n)\}$, using the representation shown in Fig. 1., one should note that each of the $g_i(n)$ elements of the ZFF output signal $\{g_i(n)\}$ can be described by a linear second-order difference equation:

$$g_i(n)=h_i(n)+\frac{\alpha}{2}\circ[h_i(n-1)+h_i(n+1)]. \tag{15}$$

As for every ZFF specified by the number i, it is required that:

$$g_i(n)=0, \qquad \text{for } n=1,2,\cdots,i \tag{16}$$

hence substituting (15) into (16) yields

$$h_i(n)+\frac{\alpha}{2}\circ[h_i(n-1)+h_i(n+1)]=0, \qquad \text{for } n=1,2,\cdots,i. \tag{17}$$

[1]A research report unpublished.

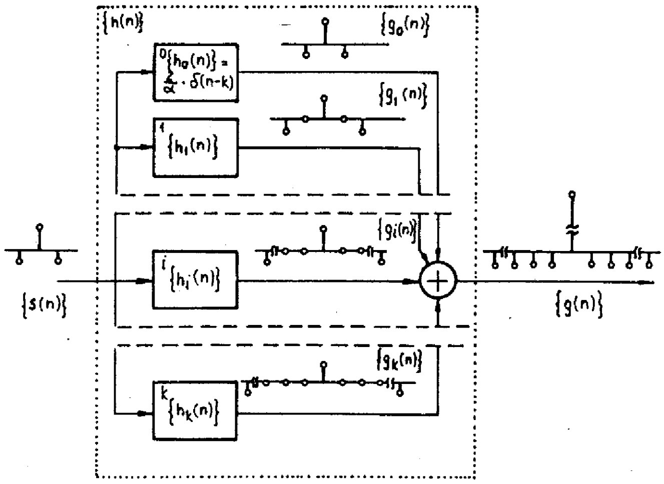

Fig. 1. SRF as parallel combination of ZFF's.

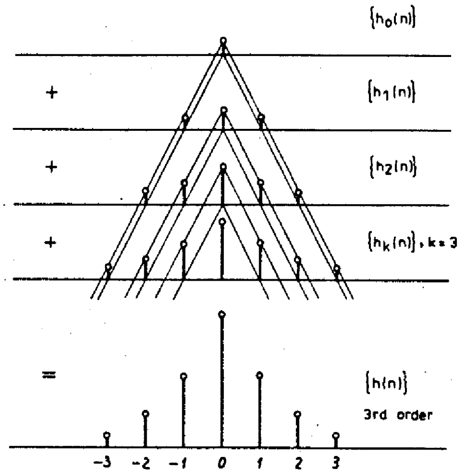

Fig. 2. Composition of SRF impulse response $\{h(n)\}$.

Equation (17) is a linear second-order homogeneous difference equation, which, with the following initial conditions:

$$h_i(i+1)=0 \tag{18}$$

and

$$\left|h_i(i)\circ\frac{\alpha}{2}\right|=1 \tag{19}$$

describes the impulse response $\{h_i(n)\}$ of every ZFF.

The first of the above conditions determines the length of each $\{h_i(n)\}$, while the second weights them in such a way that the absolute values of sidelobe amplitudes at the output of all ZFF's are equal. In our case, for the sake of convenience, the values of ∓ 1 were chosen for the sidelobe amplitudes. Substituting (18) and (19) into (17) the element $h_i(i-1)$ is obtained and consequently $h_i(i-2), h_i(i-3),\cdots$ etc., can be also computed. This means that the ith,$(i-1)$th,$\cdots$,etc., elements of every $\{h_i(n)\}$ impulse response, irrespective of the value of i, have exactly the same absolute values. Therefore, it is sufficient to solve (17) for the "longest" $\{h_i(n)\}$ i.e., for $\{h_k(n)\}$. Solutions for $h_i(n)$, for $i<k$ are contained in $\{h_k(n)\}$, which can be expressed as

$$h_{i-1}(n-1)=\operatorname{sgn}(-\alpha)\circ h_i(n), \qquad \text{for}\begin{cases}n=1,\cdots,i\\ i=0,\cdots,k.\end{cases} \tag{20}$$

Hence, the SRF impulse response $\{h(n)\}$ can be easily composed from the $\{h_k(n)\}$. A simple illustration of third-order $\{h(n)\}$ composition is depicted in Fig. 2.

The solution for the $\{h_k(n)\}$ can be found, step by step, using (17)–(19) or by solving the characteristic quadratic equation of (17), which in this case is given as [6]

$$x^2 + \frac{2}{\alpha} \circ x + 1 = 0. \qquad (21)$$

The solution for $|\alpha| \leqslant 1$ includes the two following cases:

1) Two real roots x_1, x_2, for $|\alpha| < 1$

$$h_k(n) = c_1 \circ x_1^n + c_2 \circ x_2^n. \qquad (22)$$

2) Double real root $x_{1,2}$, for $\alpha = \mp 1$

$$h_k(n) = c_1 \circ x_{1,2}^n + nc_2 \circ x_{1,2}^n \qquad (23)$$

where c_1 and c_2 denote the constants, which are to be computed by substituting (18) and (19) into (22) or (23).

Admittedly, from the standpoint of computational complexity the use of formulas (9) up to (14) and the above method of $\{h(n)\}$ computation are more or less equivalent. However, the latter offers, in addition, an insight into the structure of the SRF sought, showing it as the parallel combination of well-known ZFF's.

IV. Noise Considerations

As mentioned in Section I, the SRF is not matched to the input signal and therefore some degradation in signal-to-noise ratio, relative to a matched filter, must be expected. It is assumed here, that the SRF input signal is embedded in the additive white Gaussian noise. Then, the measure of the signal-to-noise degradation is defined as the filter output signal-to-noise ratio—SNR_o, assuming that the input signal-to-noise ratio equals 0, [dB], i.e., that both the input noise power density and the input signal energy are equal to 1, [4]. In the case of the SRF discussed the output signal-to-noise ratio is expressed as

$$\text{SNR}_o = 10\log \frac{g^2(0)}{h^2(0) + 2\sum_{j=1}^{k} h^2(j)} \qquad (24)$$

provided that

$$s^2(0) + 2s^2(1) = 1.$$

The value of $\lim \text{SNR}_o$, for $k \to \infty$, shows the tendency of the SNR_o for "long" filters, whose impulse response consists of a large number of elements. As shown in [10], for input signals which satisfy $|\alpha| < 1$,

$$\lim_{k \to \infty} \text{SNR}_o = 10\log\left[(1-\alpha^2)^{3/2}\left(1+\frac{\alpha^2}{2}\right)^{-1}\right]. \qquad (25)$$

In the limiting case of $\alpha = \mp 1$, one can derive

$$\text{SNR}_o = 10\log \frac{40(k+1)}{(2k+3)(3k^3+15k^2+28k+20)} \qquad (26)$$

which immediately gives

$$\lim_{k\to\infty} \text{SNR}_o = -\infty, \qquad \text{for } \alpha = \mp 1. \qquad (27)$$

Though not proved, numerical computations of several examples indicate, that for those $\{s(n)\}$ which satisfy $|\alpha| > 1$ $\lim \text{SNR}_o$, for $k \to \infty$, also equals $-\infty$. These results justify our assumption that the method of filtering presented could be of practical interest for those input signals which satisfy condition (8). The degradation of the signal-to-noise ratio, given by SNR_o, can be considered as the price which is to be paid for the "profit" expressed as sidelobe reduction, namely the PSLR. Thus the tradeoff between the SNR_o and the PSLR defines, in a specific way, the quality of the SRF.

V. Applications

The effectiveness of the filtering algorithm presented has been studied, among others, for two specific cases of the input signal $\{s(n)\}$.

1) For $\{s(n)\}$ with $\alpha = \mp 1$, i.e.,

$$\{s_1(n)\} = \{\mp 0.5, 1, \mp 0.5\}. \qquad (28)$$

2) For a "short" H-S, namely $\{1, 2^{-1/2}, 1, -(2)^{1/2}\}$, for which the ACF equals

$$\{s_2(n)\} = \{-(2)^{1/2}, 0, 0, 4.5, 0, 0, -(2)^{1/2}\}. \qquad (29)$$

The first of these signals is the ACF of the even pair of impulses, $(\alpha = 1)$, or the odd pair of impulses, $(\alpha = -1)$ [2]. This input signal is of particular interest because it represents the limiting case for which the SRF discussed seems still valuable in practice. The second of the signals considered is the ACF of a H-S with real elements, specified by $\alpha \cong 0.6285$.

For input signals $\{s_i(n)\}$ with $\alpha = \mp 1$ the formula for the SRF impulse response $\{h(n)\}$ has a particularly simple form and after some transformations can be derived from (9) to (12) as

$$\{h(n)\} = 1/2 \sum_{j=-k}^{k} (-1)^{|j|}(k-|j|+1) \cdot (k-|j|+2) \circ \delta(n-j) \qquad (30)$$

where $(-1)^{|j|}$ appears for $\alpha = 1$ and should be omitted for $\alpha = -1$.

For signals with $|\alpha| < 1$, $\{h(n)\}$ can be also derived analytically, [10]. However, the respective formulas have rather complicated forms and from the practical viewpoint (9)–(12) have been found to be more convenient.

For signals $\{s(n)\}$ which satisfy (8) the main performance index of the SRF, i.e., the PSLR is given as follows:

$$\text{PSLR} = -20\log\left[2\sum_{i=1}^{k+1} T_i(1/\alpha)\right] \qquad (31)$$

where $T_i(\ )$ denotes the ith-order Chebyschev polynomial of the first kind. For the derivation of (31) see Appendix B.

Tables I and II show the tradeoff between PSLR and SNR_o for $\{s_1(n)\}$ and $\{s_2(n)\}$ given by (28) and (29).

VI. Summary

The filtering algorithm presented enables us to compute the impulse response of a SRF, intended for three-element discrete even signals, which possess the form of the Huffman sequence autocorrelation function. The SRF impulse response is determined subject to the minimax criterion, which minimizes the output signal PSLR. For those input signals, whose sidelobe level is lower than half of the mainlobe level the tradeoff between the PSLR and the output signal-to-noise ratio, SNR_o is quite favorable. With the increase of the SRF length the PSLR decreases monotonically, while the SNR_o converges asymptotically to its finite value lower bound. For the limiting case of the input signal for which the sidelobe level is equal to half of the mainlobe level the degradation of SNR_o with increasing filter length, is faster than the decrease of the PSLR, which for long filters can be prohibitive. Though not proved, numerical computations indicate that for those input signals whose sidelobe level exceeds half the mainlobe level, the SNR_o deteriorates seriously with increasing filter length. For that reason, the possibility of a significant improvement in the PSLR is rather limited in these cases. It is worth mentioning, that the SRF gives superior figures for both performance indexes, i.e., PSLR and SNR_o, compared to the ZFF of the same length. It is believed that the filtering algorithm presented here can be successfully extended to input signals with more than two sidelobes. The proof of Lemma 1, (Appendix A), indicates that for an input signal for an arbitrary length, being an even sequence, the SRF impulse response subject to criterion (6) should be an even sequence too. Using the approach discussed here the SRF impulse response for 13-bit Barker code ACF was determined. The 37-element impulse response, which can be represented as the parallel combination of appropriate ZFF impulse responses, reduces input signal sidelobes to ca. -64 dB, with a drop in the SNR_o of less than 0.9 dB. Possible applications of this algorithm to a more general class of signals are being studied.

Appendix A

The purpose of this appendix is to prove, that if the minimum PSLR is to be attained, then the fact that the input signal $\{s(n)\}$ is an even sequence implies, that the SRF impulse response $\{h(n)\}$ must be an even sequence too.

Let k_1, k_2 be arbitrarily fixed natural numbers and let $k = \max(k_1, k_2)$. Let us denote by H the set of all functions $\{h(n)\}$ of integer argument n, such that

$$h(0) = \beta_o, \quad h(n) = 0, \qquad \text{for } n \in (-\infty, k_1) \cup (k_2, \infty) \tag{A.1}$$

where $\beta_0 \neq 0$ is an arbitrarily fixed real number.

Let us denote by H_e and H_o the sets of all even and odd functions $\{h_e(n)\}$ and $\{h_o(n)\}$, respectively, which take the value of 0 for $n = \mp(k+1), \mp(k+2), \cdots$. In addition assume that $h_e(0) = \beta_o$ for any $\{h_e(n)\} \in H_e$. Then any $\{h_e(n)\} \in H$ can uniquely shown as

$$\{h(n)\} = \{h_e(n)\} + \{h_o(n)\}. \tag{A.2}$$

Now, let H_e' be the set of all functions $\{h_e(n)\} \in H_e$ which are the even parts of $\{h(n)\} \in H$. Hence, we have

$$H_e' \subset H_e \text{ and } H_e' = H_e \quad \text{only if } k_1 = k_2. \tag{A.3}$$

Substitution of (A.2) into (3) yields

$$\{g(n)\} = \{g_e(n)\} + \{g_o(n)\} \tag{A.4}$$

where

$$\{g_e(n)\} = \{s(n)\} * \{h_e(n)\}$$
$$\{g_o(n)\} = \{s(n)\} * \{h_o(n)\}.$$

Now consider the following lemma.

Lemma 1: We have the following:

$$\inf_{\{h_e(n)\} \in H_e} \left[\frac{\max(|g_e(n)|,\ n \neq 0)}{|g_e(0)|} \right] \leqslant \inf_{\{h(n)\} \in H} \left[\frac{\max(|g(n)|,\ n \neq 0)}{|g(0)|} \right]. \tag{A.5}$$

Proof: From (A.4) we have

$$\{g(-n)\} = \{g_e(n)\} - \{g_o(n)\} \tag{A.6}$$

(A.4) and (A.6) imply that

$$|g_e(n)| \leqslant \max(|g(-n)|, |g(n)|), \qquad \text{for } n = 1, \cdots, k \tag{A.7}$$

which yields

$$\max_{n \neq 0} |g_e(n)| \leqslant \max_{n \neq 0} |g(n)|. \tag{A.8}$$

From (A.3) it follows that

$$\inf_{\{h_e(n)\} \in H_e} \left[\frac{\max(|g_e(n)|,\ n \neq 0)}{|g_e(0)|} \right] \leqslant \inf_{\{h_e(n)\} \in H_e'} \left[\frac{\max(|g_e(n)|,\ n \neq 0)}{|g_e(0)|} \right]. \tag{A.9}$$

If we note that $g_o(0) = 0$ and $g_e(0) = g(0)$, then substituting (A.8) into (A.9) we obtain (A.5), which completes the proof.

The thesis of *Lemma 1* implies that the value of the infimum in (6) for the SRF impulse response $\{h(n)\}$, being an even function does not exceed the value of the infimum for $\{h(n)\}$, being an arbitrary function. It is worth mentioning, that this thesis is satisfied not only in the case of input signals given by (1), but also by any $\{s(n)\}$, which is an even sequence of arbitrary but finite length.

Appendix B

The purpose of this appendix is to derive the formula for the PSLR of the kth-order SRF.

Lemma 2: If the input signal $\{s(n)\}$ satisfies (8) then for the kth-order SRF the value I of the infimum (6) attains

$$I = \left[2 \sum_{i=1}^{k+1} T_i(1/\alpha) \right]^{-1} \tag{B.1}$$

where T_i denotes the ith-order Chebyschev polynomial of the first kind.

Proof: If (8) is satisfied, then the kth-order SRF can be shown as a parallel combination of the $k+1$ ZFF's specified by number i, $i = 0, 1, \cdots, k$. Then, for every ith ZFF one can define the mainlobe–sidelobe ratio, $MSLR_i$ as

$$MSLR_i \triangleq \frac{|g_i(0)|}{|g_i(i+1)|}. \tag{B.2}$$

As for every ZFF, by its definition, (16)

$$g_i(n) = 0, \qquad \text{for } n \neq 0 \text{ and } n \neq i+1 \tag{B.3}$$

then from (7) we have

$$|g_o(1)| = |g_1(2)| = \cdots |g_i(i+1)| = \cdots |g_k(k+1)|. \tag{B.4}$$

Hence, the MSLR for the kth-order SRF, being a parallel combination of $k+1$ ZFF's, equals

$$MSLR = \sum_{i=0}^{k} MSLR_i. \tag{B.5}$$

Hence, the value I of the infimum (6) equals

$$I = (MSLR)^{-1}. \tag{B.6}$$

Now, consider the ith-order ZFF. Taking into account that the ZFF output signal $\{g_i(n)\}$ is an even sequence, (15) with condi-

TABLE I
TRADEOFF BETWEEN SRF OUTPUT PLSR AND SNR_o FOR $\{s_i(n)\}$

$\{s_1(n)\} = \{\mp 0.5, 1, \mp 0.5\}$		
k	PSLR [dB]	SNR_o [dB]
0	−6	−1.76
1	−12	−6.15
2	−15.6	−9.70
3	−18	−12.55
4	−20	−14.91
.	.	.
∞	−∞	−∞

TABLE II
TRADEOFF BETWEEN SRF OUTPUT PSLR AND SNR_o FOR $\{s_2(n)\}$

$\{s_2(n)\} = \{-\sqrt{2}, 4.5, -\sqrt{2}\}$		
k	PSLR [dB]	SNR_o [dB]
0	−10	−0.78
1	−21	−3.47
2	−30.6	−3.96
3	−39.8	−4.04
4	−48.9	−4.05
.	.	.
∞	−∞	−4.057

tion (16) can be shown in the equivalent matrix notation:

$$S_i \circ H_i = G_i \tag{B.7}$$

where

$$S_i = \begin{bmatrix} 1 & \alpha & 0 & \cdots & 0 & 0 \\ \alpha/2 & 1 & \alpha/2 & \cdots & 0 & 0 \\ 0 & \alpha/2 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & \sigma/2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & \alpha/2 \end{bmatrix}$$

$$H_i = \begin{bmatrix} h_i(0) \\ h_i(1) \\ \vdots \\ h_i(i) \end{bmatrix} \qquad G_i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ g_i(i+1) \end{bmatrix}$$

and $\{h(n)\}$ is normalized in such a way that $g_i(0) = 1$. Hence, from (B.7) it follows that

$$\text{MSLR}_i = |g_i(i+1)|^{-1} = 2 \circ |\alpha \circ h_i(i)|^{-1}. \tag{B.8}$$

Now, consider the following equation:

$$S_i^* \circ H_i = G_i^* \tag{B.9}$$

where

S_i^* $(i+1)\times(i+1)$ square matrix, which is obtained by removing the last row in S_i,

G_i^* $(i+1)$-element vector, which is obtained by removing the last element in G_i.

Using the Cramer formulas, we have from (B.7) and (B.9),

$$h_i(i) = \det S_{i-1}^* / \det S_i^* \tag{B.10}$$

where S_{i-1}^*-denotes the matrix S_i^* with the last column replaced by the vector G_i^*.

Now, let us introduce the $v \times v$ square matrix C_v,

$$C_v \circ \triangleq \begin{bmatrix} x & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2x & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2x & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 2x & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2x \end{bmatrix} \tag{B.11}$$

which has the property:

$$\det C_v = T_v(x), \qquad v = 1, 2, \cdots \tag{B.12}$$

where $T_v(x)$ denotes the vth-order Chebyschev polynomial of the first kind.

Routine transformations show that

$$\det S_{i-1}^* = (\alpha/2)^i \text{ and } \det S_i^* = 2(\alpha/2)^{i+1} \circ T_{i+1}(1/\alpha). \tag{B.13}$$

Substituting (B.13) into (B.10) we obtain $h_i(i)$, which consequently substituted into (B.8) gives the MSLR_i. Then, substituting MSLR_i into (B.5) and using (B.6) we obtain (B.1), which completes the proof. With reference to the definition of PSLR, (2), we find the following:

$$\text{PSLR} = 20 \log I. \tag{B.14}$$

Hence, substituting (B.5) we have

$$\text{PSLR} = -20 \log \left[2 \sum_{i=1}^{k+1} T_i(1/\alpha) \right] \tag{B.15}$$

which was to be derived.

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