COMPEL—The International Journal for Computation and Mathematics in Electrical and Electronic Engineering
Vol. 7, No. 3, 167–177 © 1988 BOOLE PRESS LIMITED

ON A CERTAIN DISCRETE MINIMAX PROBLEM IN DECONVOLUTION-INVERSE FILTERING*

H. UGOWSKI and A. DYKA
Department of Mathematics and Institute of Telecommunications, Technical University of Gdańsk, 80–952 Gdańsk, Poland

ABSTRACT: A solution of a discrete minimax problem is derived, using the standard methods of approximation theory. It represents a problem of the deconvolution-inverse filtering of a three-element discrete even signal, subject to a constraint on the output signal sidelobe level. Consequently, properties of this solution, in particular the asymptotic behaviour of the parameter, which represents the degradation of signal-to-noise ratio due to the filtering are examined.

1. FORMULATION OF THE MINIMAX PROBLEM AND THE MAIN RESULTS

Prior to the formulation of the mathematical problem, its physical interpretation is to be defined. Our considerations concern the special case of the problem of linear deconvolution-inverse filtering [6].

By definition an ideal linear deconvolution-inverse filter is a circuit or a computational algorithm, which for a given input signal responds with Dirac's delta distribution. In the case of a discrete sampled input signal, which we shall investigate, the output signal takes a form of a Kronecker's delta. As ideal deconvolution-inverse filters are not feasible, one has to accept the approximate solution, in a certain sense. The inevitable approximation error manifests itself in the appearance of undesirable deltas in the output signal. These additional deltas are usually referred to as the sidelobes.

Our objective here is to find the impulse response of an approximate deconvolution-inverse filter, for an input signal being a three-element discrete even sequence, subject to a constraint on the output signal sidelobe level. The rigorous formulation and the solution to this problem will be now presented.

Let us introduce a signal

$$s(n) = a_n\ ,\ n = 0, \pm 1, \pm 2, \ldots ,$$

where

$$a_0 = 1,\ a_{-1} = a_1 = a,\ a_n = 0,\ n = \pm 2, \pm 3, \ldots$$

and $a \neq 0$ is a real constant. The function $s(n)$ represents a special type of signal

* This work has been carried out with the support of the C.P.B.P. 02.16.4.2 research project.

which is actually the autocorrelation function of the Huffman code (e.g. [2]). The impulse response required is denoted as

$$h(n) = h_n,\ n = 0, \pm 1, \pm 2, \ldots,$$

where

$$h_0 = 1,\ h_{-n} = h_n \quad \text{for } n = 1, \ldots, k,\ h_n = 0 \quad \text{for } n = \pm(k+1), \pm(k+2), \ldots$$

and k is an arbitrarily fixed positive integer. The output signal is given by the convolution

$$g(n) = s(n) * h(n) = \sum_i s(i)h(n-i) = g_n .$$

Hence,

$$g_{-n} = g_n = ah_{n-1} + h_n + ah_{n+1},\ n = 0, 1, \ldots, k-1, \tag{1.1}$$

$$g_{-k-1} = g_{k+1} = ah_k,\ g_{-k} = g_k = ah_{k-1} + h_k, \tag{1.2}$$

$$g_n = 0,\ n = \pm(k+2), \pm(k+3), \ldots \tag{1.3}$$

The function $g(n)$ should be as "close" as possible to the discrete function consisting of the single element g_0. The criterion of this closeness is defined here as the following minimax problem (M_h).

(M_h) Determine numbers $\bar{h}_1, \ldots, \bar{h}_k$ which realize

$$\inf_h \max_{n \neq 0} |g_n g_0^{-1}|,\ h = (h_1, \ldots, h_k).$$

The main results of the present paper are formulated in the following theorem.

THEOREM 1: Assume that a is a real parameter such that $0 < |a| \leq 1/2$ and denote $e = \operatorname{sign}(-a)$. Then the following assertions hold:

(A) There exists a unique solution $\bar{h}_1, \ldots, \bar{h}_k$ to the problem (M_h).

(B) The above solution is defined by the formula

$$\bar{h}_n = \bar{p}_n \bar{p}_0^{-1},\ n = 1, \ldots, k, \tag{1.4}$$

where $\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_k$ is a solution to the system of equations

$$ap_n + p_{n+1} + ap_{n+2} = e^{k+n},\ n = 0, 1, \ldots, k-2, \tag{1.5}$$

$$ap_{k-1} + p_k = e,\ ap_k = 1. \tag{1.6}$$

Consequently we have

$$\bar{p}_k = a^{-1},\ \bar{p}_{k-1} = ea^{-1} - a^{-2},$$

$$\bar{p}_n = a^{-1}e^{k+n} - a^{-1}\bar{p}_{n+1} - \bar{p}_{n+2},\ n = k-2, \ldots, 1, 0.$$

(C) The solution $\bar{h}_1, \ldots, \bar{h}_k$ and the value $\bar{g}_n$ ($n = 0, 1, \ldots, k+1$) of the function g_n for this solution have the following properties:

1° $$1 > |\bar{h}_1| > |\bar{h}_2| > \cdots > |\bar{h}_k| > 0, \text{ sign } \bar{h}_n = e^n, \ n = 1, \ldots, k \ ;$$

2° $$0 < \bar{g}_0 < 1 \ ;$$

3° $$\text{sign } \bar{g}_n = -e^n, \ \bar{g}_n = e^{n+k+1}\bar{g}_{k+1} = -e^n |a| \, |\bar{h}_k| \ ;$$

4° $$\inf_h \max_{n \neq 0} |g_n g_0^{-1}| = |a\bar{h}_k| \bar{g}_0^{-1} = |a\bar{h}_k|(1 + 2a\bar{h}_1)^{-1} \ ;^*$$

5° the function
$$D_k(a) = \bar{g}_0^2 \left(1 + 2 \sum_{i=1}^{k} \bar{h}_i^2\right)^{-1} \tag{1.7}$$

has the following properties:

$$0 < D_k(a) < 1 \ , \tag{1.8}$$

$$\lim_{k \to \infty} D_k(a) = (1 - 4a^2)^{3/2} \ . \tag{1.9}$$

The above theorem will be proved in Section 3. The proof is based on two auxiliary theorems and the fundamental theory of recurrence equations [4], Cramer's rule and geometric sequences. The auxiliary theorems are formulated and proved in Section 2. The first auxiliary theorem (Theorem 2) gives necessary and sufficient conditions for the parameter a, which assure the existence of a unique solution to the problem (M_h). Moreover, the analytic formulae for this solution are given. To prove Theorem 2 we replace the nonlinear problem (M_h) by a certain linear minimax problem (M_H). Consequently, the problem (M_H) is solved with the aid of the standard method of linear approximation theory [1]. The second auxiliary theorem (Theorem 3) retains assumptions of Theorem 2 and gives another effective method of solving the problem (M_h). The proof of Theorem 3 is simple and it consists of using some statements from the proof of Theorem 2.

Now we present the interpretation of the function $D_k(a)$. In terms of signal theory it represents the degradation of signal-to-noise ratio due to the filtering. If the signal-to-noise ratio is normalized with respect to the matched filter (which yields maximum of signal-to-noise ratio), then it takes values between 1 and 0 [4]. Values of $D_k(a)$ below 0.1 indicate that the degradation of signal-to-noise ratio is too great for most practical applications. The value of $\lim_{k\to\infty} D_k(a)$ shows the tendency in behaviour of signal-to-noise-ratio for "long" filters, whose impulse responses consist of a large number of elements. Though not proved, numerical computations of several examples indicate that for $|a| > \frac{1}{2}$ we have

$$\lim_{k \to \infty} D_k(a) = 0 \ .$$

* In [3] it is proved that

$$\inf_h \max_{n \neq 0} |g_n g_0^{-1}| = 2 \sum_{n=1}^{k+1} T_n\left(\frac{1}{2a}\right),$$

where T_n is the nth order Tschebyschev polynomial of the first kind.

Hence, from eq. (1.9), it follows that the above filtering method can be of practical interest mainly for those input signals which satisfy the condition $0<|a|<\frac{1}{2}$.

2. THE AUXILIARY THEOREMS

First we have to introduce some notation and assumptions. We need the following nth order determinant

$$V_n(a, b) = \begin{vmatrix} b & a & 0 & 0 & \dots & & & 0 \\ a & 1 & a & 0 & \dots & & & 0 \\ 0 & a & 1 & a & 0 & \dots & & 0 \\ \dots & & & & & & & \dots \\ 0 & \dots & & & 0 & a & 1 & a \\ 0 & \dots & & & 0 & 0 & a & 1 \end{vmatrix} \tag{2.1}$$

It is easy to see that

$$V_{n+2}(a, b) = V_{n+1}(a, b) - a^2 V_n(a, b), n = 0, 1, 2, \dots , \tag{2.2}$$

where

$$V_0(a, b) = 1, V_1(a, b) = b . \tag{2.3}$$

We now introduce the new variables

$$H_i = h_i g_0^{-1} = h_i(1 + 2ah_1)^{-1}, i = 1, \dots , k . \tag{2.4}$$

Then, from eqs. (1.1), (1.2), the functions $g_n g_0^{-1}$, $n = 1, \dots , k + 1$ become

$$G_1 = g_1 g_0^{-1} = a + (1 - 2a^2)H_1 + aH_2 , \tag{2.5}$$

$$G_n = g_n g_0^{-1} = aH_{n-1} + H_n + aH_{n+1}, n = 2, \dots , k - 1 , \tag{2.6}$$

$$G_k = g_k g_0^{-1} = aH_{k-1} + H_k, G_{k+1} = g_{k+1} g_0^{-1} = aH_k . \tag{2.7}$$

Consequently, the problem (M_h) is equivalent to the following problem (M_H).

(M_H) Determine numbers $\bar{H}_1, \dots , \bar{H}_k$ which realize

$$\inf_H \max_{n\neq 0} |G_n|, H = (H_1, \dots , H_k) .$$

We solve the problem (M_H) applying the standard method of linear approximation theory (see [1], Sections 2.3 and 2.4). For this purpose we introduce the following system of equations

$$(1 - 2a^2)d_1 + ad_2 = 0 , \tag{2.8}$$

$$ad_n + d_{n+1} + ad_{n+2} = 0, n = 1, 2, \dots , k - 1 . \tag{2.9}$$

Denote by A_n $(n = 1, \ldots, k+1)$ the vector of the space R^k whose components are the elements of the nth column of the matrix of the system (2.8), (2.9). Assume that $a \neq 0$ is such a real number that

$$V_n(a, 1-2a^2) \neq 0, n = 1, \ldots, k . \tag{2.10}$$

Denoting by B_n $(n = 1, \ldots, k+1)$ the kth order determinant whose rows are all vectors A_i except for A_n, one can easily find that

$$B_n = a^{k+1-n} V_{n-1}(a, 1-2a^2), n = 1, \ldots, k+1 .$$

Hence, it follows that in eq. (2.10) it is equivalent to the assumption that each system of k vectors selected from $\{A_1, \ldots, A_{k+1}\}$ is linearly independent. Consequently, there exists a unique solution to the problem (M_H). Moreover, if

$$d_1, \ldots, d_{k+1} \tag{2.11}$$

is a nonzero solution to the system (2.8), (2.9), then

$$d_n \neq 0, n = 1, \ldots, k+1 \tag{2.12}$$

and the set of all solutions to the above problem is a one dimensional linear subspace of R^{k+1}. We now determine a solution (2.11) such that $d_1 = 1$. It obviously follows from eqs. (2.8) and (2.9) that this solution is given by the formula

$$d_1 = 1, d_2 = 2a - a^{-1}, d_n = -d_{n-2} - a^{-1} d_{n-1}, n = 3, \ldots, k+1 . \tag{2.13}$$

Consequently, this solution satisfies in eq. (2.12). Now we use the standard theory that the variables minimize $\max_{n \neq 0} |G_n|$ if, and only if, their values give

$$G_n = K e_n, n = 1, \ldots, k+1 \tag{2.14}$$

for some constant K, where

$$e_n = \operatorname{sign} d_n, n = 1, \ldots, k+1 \tag{2.15}$$

and d_n are defined by eqs. (2.13). Then we have

$$\inf_H \max_{n \neq 0} |G_n| = |K| . \tag{2.16}$$

Taking into account that system (2.13) satisfies the system of equations (2.8), (2.9) and using relations (2.5)–(2.7), (2.14), (2.15) one can find that

$$K = \Big(\sum_{n=1}^{k+1} d_n G_n\Big)\Big(\sum_{n=1}^{k+1} |d_n|\Big)^{-1} = a\Big(\sum_{n=1}^{k+1} |d_n|\Big)^{-1} \tag{2.17}$$

Eq. (2.14) for $n = 2, \ldots, k+1$ and relations (2.6), (2.7) enable us to obtain the following recurrence formula for the solution to the problem (M_H)

$$\bar{H}_k = Ka^{-1}e_{k+1},\ \bar{H}_{k-1} = Ka^{-1}e_k - a^{-1}\bar{H}_k\,,$$
$$\bar{H}_n = Ka^{-1}e_{n+1} - a^{-1}\bar{H}_{n+1} - \bar{H}_{n+2},\ n = k-2, k-3, \dots, 1\,. \tag{2.18}$$

If

$$\bar{H}_1 \neq (2a)^{-1}\,, \tag{2.19}$$

then relations (2.4) yield the following unique solution to the problem (M_h)

$$\bar{h}_i = \bar{H}_i(1 - 2a\bar{H}_1)^{-1},\ i = 1, \dots, k\,. \tag{2.20}$$

In order to formulate a condition which implies in eq. (2.19) we determine $\bar{H}_1$ applying Cramer's rule to system (2.14) with $n = 2, \dots, k+1$. Consequently, the following condition

$$\sum_{i=1}^{k} (-1)^{i+1} e_{i+1} a^{-i+1} V_{i-1}(a, 1) \neq (2K)^{-1} \tag{2.21}$$

is equivalent to in eq. (2.19). At the same time we have proved the following auxiliary theorem given below.

THEOREM 2: Let $a \neq 0$ be an arbitrarily fixed real number satisfying conditions (2.10) and (2.21), where formulae (2.17), (2.13) and (2.15) are used. Then there exists a unique solution $\bar{h}_1, \dots, \bar{h}_k$ to the problem (M_h). Moreover,

$$\inf_h \max_{n\neq 0} |g_n g_0^{-1}| = |K|$$

and the above solution is given by eqs. (2.20), (2.18).

REMARK 1: It can be shown that the assumptions of Theorem 2 are necessary for the existence of a unique solution to the problem (M_h).

REMARK 2: If $a \neq 0$ satisfies in eq. (2.10) while at the same time it does not satisfy in eq. (2.19), then no solution exists to the problem (M_h).

Now we formulate and prove the following auxiliary theorem.

THEOREM 3: Let the assumptions of Theorem 2 be satisfied. Then the unique solution to the problem (M_h) is defined by the formula

$$\bar{h}_n = \bar{p}_n \bar{p}_0^{-1},\ n = 1, \dots, k\,, \tag{2.22}$$

where $\bar{p}_0, \bar{p}_1, \dots, \bar{p}_k$ is the solution to the system of equations

$$ap_n + p_{n+1} + ap_{n+2} = e_{n+1}e_{k+1},\ n = 0, 1, \dots, k-2\,, \tag{2.23}$$

$$ap_{k-1} + p_k = e_k e_{k+1},\ ap_k = 1\,. \tag{2.24}$$

Consequently, we have

$$\bar{p}_k = a^{-1},\ \bar{p}_{k-1} = e_k e_{k+1} a^{-1} - a^{-2},$$

$$\bar{p}_n = a^{-1} e_{n+1} e_{k+1} - a^{-1} \bar{p}_{n+1} - \bar{p}_{n+2},\ n = k-2, \ldots, 1, 0.$$

Proof. It follows from the considerations which precede Theorem 2 that the unique solutions $\bar{H}_1, \ldots, \bar{H}_k$ and $\bar{h}_1, \ldots, \bar{h}_k$ to the problems (M_H) and (M_h) exist, respectively. Denote by $\bar{G}_n$ and $\bar{g}_n$ the values of functions G_n and g_n for these solutions, respectively. From eq. (2.14) we have

$$\bar{G}_n = e_n e_{k+1} \bar{G}_{k+1},\ n = 1, \ldots, k+1.$$

Hence, from eqs. (1.1), (1.2) and eqs. (2.5)–(2.7), it follows that

$$\bar{g}_n = e_n e_{k+1} \bar{g}_{k+1},\ n = 1, \ldots, k+1. \tag{2.25}$$

Multiplying each of the last equalities by $(a\bar{h}_k)^{-1}$ and denoting

$$\bar{p}_0 = (a\bar{h}_k)^{-1},\ \bar{p}_n = \bar{h}_n \bar{p}_0,\ n = 1, \ldots, k$$

we conclude that the above system $\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_k$ is the unique solution to the system (2.23), (2.24) and that eq. (2.22) is satisfied. This completes the proof.

3. PROOF OF THEOREM 1

First we show that the assumptions of Theorem 2 are satisfied. We begin with the case $0 < |a| < \frac{1}{2}$. Applying the recurrence equations theory to eq. (2.2) [5] one obtains the following formula

$$V_n(a, b) = C_1 r_1^n + C_2 r_2^n,\ n = 0, 1, 2, \ldots, \tag{3.1}$$

where C_1, C_2 are any real constants depending only on a and b, and r_1, r_2 are the roots of the characteristic equation

$$r^2 - r + a^2 = 0,$$

which implies that

$$r_1 = \tfrac{1}{2}[1 - (1 - 4a^2)^{1/2}],\ r_2 = \tfrac{1}{2}[1 + (1 - 4a^2)^{1/2}]. \tag{3.2}$$

Constants C_1, C_2 are determined by conditions (2.3) which implies the following form for formula (3.1)

$$V_n(a, b) = (1 - 4a^2)^{-1/2}[(r_2 - b) r_1^n + (b - r_1) r_2^n],\ n = 0, 1, 2, \ldots \tag{3.3}$$

Making use of the last formula one can find that

$$V_n(a, \tfrac{1}{2}) = \tfrac{1}{2}(r_1^n + r_2^n) > 0,\ n = 0, 1, 2, \ldots$$

and

$$\frac{\partial V_n(a, b)}{\partial b} = (1 - 4a^2)^{-1/2}(r_2^n - r_1^n) \geq 0,\ n = 0, 1, 2, \ldots$$

Hence it follows that

$$V_n(a, b) > 0,\ n = 0, 1, 2, \ldots, b \geq \tfrac{1}{2}\,,$$

and consequently

$$V_n(a, 1 - 2a^2) > 0,\ n = 0, 1, 2, \ldots,$$

which completes the proof of (2.10).

Now we derive a new formula for the solution (2.13) to the system (2.8), (2.9) using the recurrence equations theory. Then, we have the following solution to system (2.9)

$$d_n = C_1 s_1^n + C_2 s_2^n,\ n = 1, \ldots, k + 1\,, \tag{3.4}$$

where C_1, C_2 are constants and

$$s_1 = (2a)^{-1}[-1 - (1 - 4a^2)^{1/2}],\ s_2 = (2a)^{-1}[-1 + (1 - 4a^2)^{1/2}]\,. \tag{3.5}$$

The condition $d_1 = 1$ and eq. (2.8) imply the following form of the formula (3.4)

$$d_n = -a(s_1^n + s_2^n),\ n = 1, 2, \ldots, k + 1\,. \tag{3.6}$$

Of course, systems (3.6) and (2.13) are identical. Hence, from eqs. (2.15) and (3.5), we have

$$e_n = e^{n+1},\ e = \operatorname{sign}(-a),\ n = 1, \ldots, k + 1\,. \tag{3.7}$$

Relations (2.17), (3.5), (3.7) imply that

$$K = a^{-1}(1 - 2|a|)[(1 - |s_1|)(1 - |s_1|^{k+1}) + (1 - |s_2|)(1 - |s_2|)^{k+1}]^{-1}\,. \tag{3.8}$$

Using the formula

$$V_n(a, 1) = (r_2 - r_1)^{-1}(r_2^{n+1} - r_1^{n+1}),\ n = 0, 1, 2, \ldots \tag{3.9}$$

(following from eqs. (3.3), (3.2)) and eqs. (2.17), (3.5)–(3.7), (3.2) we conclude that

$$\sum_{i=1}^{k} (-1)^{i+1} e_{i+1} a^{-i+1} V_{i-1}(a, 1) - (2K)^{-1}$$

$$= -\frac{e}{2}\left[2 \sum_{i=1}^{k} \frac{|s_1|^i - |s_2|^i}{|s_1| - |s_2|} + \sum_{i=1}^{k+1} (|s_1|^i + |s_2|^i)\right]$$

$$= \frac{e}{2(|s_2| - |s_1|)}\left[\sum_{i=1}^{k} (|s_1|^i - |s_2|^i) + \sum_{i=1}^{k+1} (|s_1|^i - |s_2|^i)\right],$$

which completes the proof of condition (2.21).

We now consider the case $|a| = \frac{1}{2}$. Having proceeded as in the previous case we obtain the following formulae

$$V_n(a, b) = [1 + n(2b - 1)]2^{-n}, n = 0, 1, 2, \ldots, \tag{3.10}$$

$$d_n = e^{n+1}, e_n = \text{sign } d_n = e^{n+1}, e = \text{sign}(-a), n = 1, \ldots, k+1 . \tag{3.11}$$

Relations (3.11), (3.10) and (2.17) imply that

$$K = a(k+1)^{-1}, \sum_{i=1}^{k} (-1)^{i+1} e_{i+1} a^{-i+1} V_{i-1}(a, 1) = -ak(k+1) . \tag{3.12}$$

In virtue of eqs. (3.10), (3.12), assumptions (2.10) and (2.21) are satisfied. At the same time, according to Theorem 2, we have proved assertion (A) of Theorem 1.

Assertion (B) of Theorem 1 immediately follows from Theorem 3 and relation (3.7) (for $0 < a| \leq \frac{1}{2}$).

To prove assertion (C), 1° of Theorem 1 we determine first the solution $\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_k$ to the system (1.5), (1.6) with the aid of Cramer's rule. Then we have

$$\bar{p}_n = a^{-k-1} W_{n+1}, n = 0, 1, \ldots, k ,$$

where W_{n+1} denotes the appropriate $(k+1)$th order determinant. Hence, expanding W_{n+1} with respect to its $(n+1)$th column one obtains

$$\bar{p}_n = a^{-1} e^{k+n} \sum_{i=0}^{k-n} |a|^{-i} V_i(a, 1), n = 0, 1, \ldots, k . \tag{3.13}$$

This implies, by assertion (B), that

$$\bar{h}_n = \bar{p}_n \bar{p}_0^{-1} = e^n \sum_{i=0}^{k-n} |a|^{-i} V_i(a, 1) \left[\sum_{i=0}^{k} |a|^{-i} V_i(a, 1) \right]^{-1}, n = 1, \ldots, k \tag{3.14}$$

is the unique solution to the problem (M_h). In view of the inequality $V_i(a, 1) > 0$ formula (3.14) yields assertion (C), 1°.

Assertion (C), 2° is an immediate consequence of (C), 1° because $-1 < 2a\bar{h}_1 < 0$.

Assertion (C), 3° follows from eqs. (2.25) and (3.7) (for $0 < |a| \leq \frac{1}{2}$).

Also, assertion (C), 4°, in view of (C), 3°, holds true.

It remains to prove (C), 5°. Inequality (1.8) is an immediate consequence of eq. (1.7) and (C), 2°. To prove eq. (1.9) we use the formula

$$D_k(a) \doteq (\bar{p}_0 + 2a\bar{p}_1)^2 \left(\bar{p}_0^2 + 2 \sum_{i=1}^{k} \bar{p}_i^2 \right)^{-1} \tag{3.15}$$

following from eqs. (1.7) and (1.4). We first consider the case $0 < |a| < \frac{1}{2}$. Using relations (3.13), (3.3), (3.2) and (3.5) we get

$$|\bar{p}_i| = C\left[\frac{|s_1| - |s_1|^{k-i+2}}{1-|s_1|} - \frac{|s_2| - |s_2|^{k-i+2}}{1-|s_2|}\right], \quad i = 0, 1, \ldots, k,$$
$$|\bar{p}_0 + 2a\bar{p}_1| = C\left|\frac{|s_1| - |s_1|^{k+2}}{1-|s_1|} - \frac{|s_2| - |s_2|^{k+2}}{1-|s_2|} + 2|a|\left(\frac{|s_2| - |s_2|^{k+1}}{1-|s_2|} - \frac{|s_1| - |s_1|^{k+1}}{1-|s_1|}\right)\right| \tag{3.16}$$

where $C = (1-4a^2)^{-(1/2)}$. Hence, it follows that

$$\lim_{k\to\infty} |s_1|^{-k}|\bar{p}_0 + 2a\bar{p}_1| = \frac{C|s_1|(|s_1| - 2|a|)}{|s_1| - 1}, \tag{3.17}$$

$$\lim_{k\to\infty} |s_1|^{-k}|\bar{p}_0| = Cs_1^2(|s_1| - 1)^{-1} \tag{3.18}$$

Using (3.16) and calculating $\bar{p}_i^2$ and $\Sigma_{i=1}^k \bar{p}_i^2$ one can find that

$$\lim_{k\to\infty} s_1^{-2k} \sum_{i=1}^{k} \bar{p}_i^2 = C^2 s_1^4 (|s_1| - 1)^{-2}(s_1^2 - 1)^{-1}. \tag{3.19}$$

Relations (3.15), (3.17)–(3.19) imply that

$$\lim_{k\to\infty} D_k(a) = (|s_1| - 2|a|)^2 s_1^{-2}[1 + 2(s_1^2 - 1)^{-1}]^{-1},$$

and consequently we get eq. (1.9).

Finally, we consider the case $|a| = \frac{1}{2}$. It follows from eqs. (3.13) and (3.10) that

$$\bar{p}_i = -e^{k-i-1}(k-i+1)(k-i+2), i = 0, 1, \ldots, k.$$

Hence, from eq. (3.15), we have

$$D_k(\pm\tfrac{1}{2}) = 4(k+1)^2[(k+1)^2(k+2)^2 + 2\sum_{i=1}^{k}(k-i+1)^2(k-i+2)^2]^{-1}.$$

Calculating the last expression with the aid of the formulae

$$\sum_{i=1}^{k}(k-i+1)^2(k-i+2)^2 = \sum_{j=1}^{k} j^2(j+1)^2,$$
$$\sum_{j=1}^{k} j^2 = \frac{k}{6}(k+1)(2k+1), \quad \sum_{j=1}^{k} j^3 = \frac{1}{4}k^2(k+1)^2,$$
$$\sum_{j=1}^{k} j^4 = \frac{1}{30}k(k+1)(6k^3 + 9k^2 + k - 1)$$

we obtain

$$D_k(\pm\tfrac{1}{2}) = 60(k+1)(6k^4 + 39k^3 + 101k^2 + 124k + 60)^{-1}.$$

This implies that

$$\lim_{k\to\infty} D_k(\pm\tfrac{1}{2}) = 0$$

and consequently eq. (1.9) holds also for $|a| = \frac{1}{2}$. At the same time we have completed the proof of Theorem 1.

ACKNOWLEDGMENT

The authors greatfully acknowledge valuable comments and advice of Professor M.J.D. Powell of Cambridge University, England concerning the previous version of this paper.

REFERENCES

[1] E.W. Cheney, Introduction to Approximation Theory (McGraw Hill, New York, 1966).
[2] C.E. Cook and M. Bernfeld, Radar Signals—An Introduction to Theory and Application (Academic Press, London, 1967).
[3] A. Dyka and H. Ugowski, Minimax Sidelobe Reduction Filtering for Huffman Sequence Autocorrelation Function Type Signals, to be published in Vol. CAS-35(8) (1988) of IEEE Transactions on Circuits and Systems, U.S.A.
[4] R.J. Evans and T.E. Fortmann, Optimal resolution of rectangular pulses in noise, IEEE Trans. on AES 11 (3) (1975) 372–379.
[5] R.A. Gabel and R.A. Roberts, Signals and Linear Systems (Wiley, New York, 1973).
[6] D.G. Senmoto and S. Childers, Signal resolution via digital inverse filtering, IEEE Trans. on AES 8(5) (1972) 633–640.

ERRATUM

Due to late changes in reference numbers, some incorrect attributions were made in the paper "Results for benchmark problem 5", by A. Bossavit, in COMPEL, **7**, 1 & 2 (1988), pp. 81-88. All references to experimental data should have been to Ref. /1/ (the original paper by Davidson and Balchin). Figure 2, showing current density on the faces of the Bath cube, is from the work of Nakata et al., Ref. /5/. The author regrets these errors.

A. Bossavit