ANDRZEJ DYKA MAREK MOSZYŃSKI
Technical University of Gdańsk

OPTIMUM FIR INTEGRATOR FOR EXPONENTIAL SIGNALS

SUMMARY: The performances of FIR integrator for exponential signal are presented and the equation for optimal integrator length is given. The application of optimal integration and deconvolution techniques filtering to exponential signals is shown.

1. INTRODUCTION

A problem of considerable importance in various areas of science and technology is a retrieval of an exponential signal from noise and the identification of its parameters, i.e., the time constant and amplitude. Prior to the identification procedures the signal is to be filtered to improve the signal-to-noise ratio. The most straightforward solution for such a filter is an FIR integrator. In this paper the maximum signal-to-noise ratio approach is presented, which for a given exponential signal allows for the derivation of the optimum FIR integrator.

2. FORMULATION OF THE PROBLEM

Given a discrete exponential signal $\{s(n)\}$:

$$\{s(n)\}=A\ [1,r,r^2,\dots,r^l] \qquad (1)$$

where

A - amplitude,

$r=e^{-a}$, $a=\alpha T$, $1/\alpha$ - time constant, T - sampling rate.

We say that the length of the signal $\{s(n)\}$ equals $l+1$.

Now, assume that the FIR integrator impulse response has the following form:

$$\{h_k(n)\}=[1,1,...,1] \qquad (2)$$

The subscript k indicates that the integrator impulse response consists of k "ones", and therefore we say that the integrator length is k.

The output signal of the integrator - response to the input exponential signal (1) is as follows.

$$y(n)=s(n)*h(n)=A\begin{cases} \frac{1-r^n}{1-r} & n = 1,...,k-1 \\ \frac{1-r^k}{1-r}\, r^{n-k} & n = k,...,l+1 \\ \frac{1-r^{l+1}}{1-r} - \frac{1-r^{n+1-l}}{1-r} & n = l+2,...,k+1 \end{cases} \qquad (3)$$

One will observe that for n = k,...,l+1, i.e the steady state of the integrator response (3) the said response is an exponential signal of the same time constant as the input signal.

$$\{y_2(n)\}=A_2[1,r,r^2,...r^{l+1-k}] \qquad (4)$$

where $A_2 = A\frac{1-r^k}{1-r}$ - amplitude of the signal $\{y_2(n)\}$

In practical applications the input signal is contamined with noise. We assume that this is a zero-mean, white gaussian noise. Our objective here is to determine ths integrator length k in such a way as to obtain the possibly "best" suppression of the noise at the output of the integrator. As the criterion of the quality of noise suppression we assume the output signal to noise ratio, defined as follows:

$$SNR = \frac{E[y]}{E[s]\ \sum_2 h(n)} \qquad (5)$$

where E[x] stands for the energy of x.

One can find that the energies of s and y are given as, respectively:

$$E[s] = \sum_n s(n) = A\frac{1-r^{2(l+1)}}{1-r^2} \quad \text{oraz} \tag{6}$$

$$E[y_2] = \sum_n y_2(n) = A\left[\frac{1-r^k}{1-r}\right]^2 \frac{1-r^{2(l+2-k)}}{1-r^2} \tag{7}$$

Hence, the signal-to-noise ratio (5) equals:

$$SNR = \frac{1}{(1-r)^2} \; \frac{(1-r^k)^2}{k} \; \frac{1-r^{2(l+2-k)}}{1-r^2} \tag{8}$$

After, some routine though tedious transformations, one obtains the following condition for the maximum of signal-to-noise ratio:

$$cth(ak_{opt}/2) - cth[a(l+2-k_{opt})] = 1/ak_{opt} \tag{9}$$

Define the following normalization

$$K_{opt} = ak_{opt}, \quad L = a(l+2) \tag{10}$$

and substitute to (9). Then, one obtains:

$$\boxed{cth(K_{opt}/2) - cth(L-K_{opt}) = 1/K_{opt}} \tag{11}$$

This is a condition for the optimum integrator length which provides maximum of the signal-to-noise ratio. Unfortunately, the optimum length K_{opt} is given implicite and therefore the solution for it can be obtained in a numerical way only. The dependence of $K_{opt}(l)$ and $k_{opt}(l)$ vs signal length for various values of a are given in fig 1a and 1b,respectively. Fig. 1c, illustrates the course of the signal-to-noise vs. normalized signal length for a few time constants 1/a. In this case for every signal length l the optimum integrator length k_{opt} is computed. One can see that for low values of the integrator length the signal-to-noise ratio is poor and it increases with increasing l. For a certain threshold value of l it gets into saturation and does not increase significantly with increasing l.

Now, we present a simple algorithm which allows for determining the optimum integrator length for an input exponential signal with a given time constant 1/a. We provide approximate closed-form explicite expressions for the signal length l, integrator length k_{opt}, and the maximum of signal-to-noise ratio in the sense of definition (5) .

First we compute the normalized value of K_{opt} for $L \longrightarrow \infty$. Substitution to (11) gives:

$$\mathrm{cth}(K_{opt}/2)-1=1/K_{opt} \qquad (12)$$

which provides the solution $K_{opt}(L_\infty)=1.2564...$ .The value of $K_{opt}=1.25$ is attained for:

$$L(K_{opt}=1.25)=K_{opt}+\mathrm{arcth}[\mathrm{cth}(K_{opt}/2)-1/K_{opt}]=4.485...\approx 4.5$$

Hence, the signal length l should meet the following inequality

$$l \geq \frac{4.5}{a} \qquad (13)$$

and the optimum integrator length equals approximately

$$k_{opt} = \frac{1.25}{a} \qquad (14)$$

Consequently, using (8) one can determine the maximum of signal-to-noise ratio SNR_{max} as:

$$SNR_{max} = \lim_{l\to\infty} SNR = 0.8(1-e^{-1.25})^2 \frac{a}{(1-e^{-a})^2} \approx \frac{0.41}{a} \qquad (15)$$

which gives the following result in dBs

$$SNR_{max}[dB]=10\log(SNR_{max})=10\log(\frac{1}{a})-4, \ [dB] \qquad (16)$$

In Fig.1d the SNR vs. signal length l for the following integrator lengths $k1=k_{opt}$, $k2=0.5l$, $k3=(5/18)l$, is presented. The values mentioned above have been obtained from linear approximation of optimal integrator normalized characteristic (fig. 1b).

3.APPLICATION RESULTS

The above results were applied for the identification of an exponential signal using deconvolution filtering method[1]. The length l of the input signal was determined using formulae (13) and (14). It is notable that these formulae impose limits on the dynamic range of the input signal, which depends on both length and time constant. Hence , one can find that the required dynamic range of the signal equals

$$20\log\frac{s(0)}{s(l_{min})}=20\log\frac{A}{Ae^{-al_{min}}}=20\log(e^{al_{min}})=20\log(e^{4.5})=$$

$$=20\log(90)=39\ [dB]$$

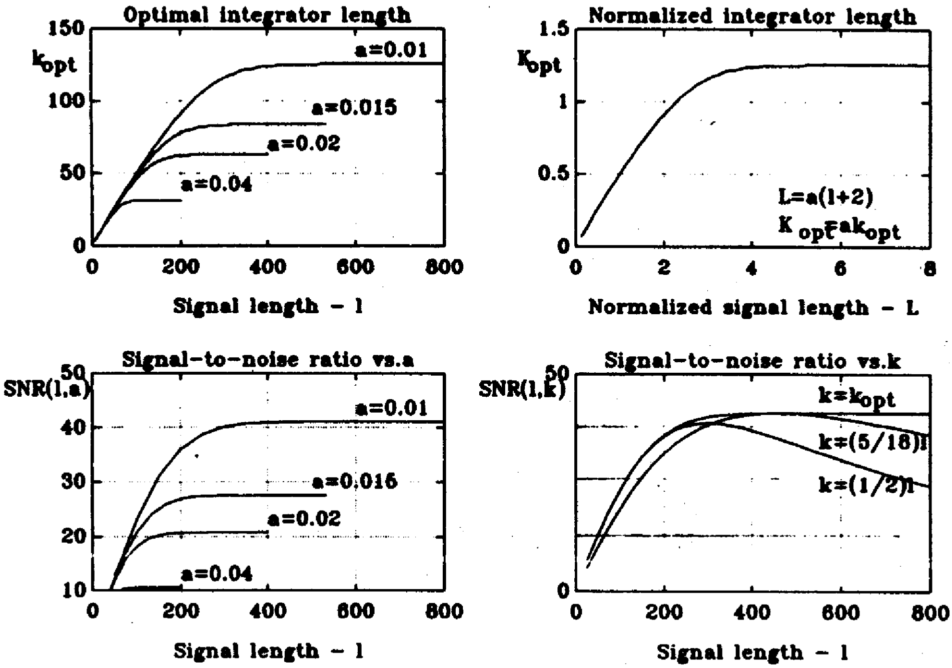

Fig 1.

a. Optimal integrator length vs. time constant 1/a.

b. Optimal integrator normalized characteristic.

c. Signal-to-noise ratio vs. time constant 1/a.

d. Signal-to-noise ratio vs. integrator length

In order to determine the optimum FIR integrator, the exponential time constant 1/a should be a priori known, which as a rule is not a case. Therefore, in the first step one should use the integrator of an arbitrary length k. As shown in Fig. 1d this length should be chosen from the interval k ∊ (5/18)l..(1/2)l. After such a preliminary integration a deconvolution filtering method for the identification of the time constant 1/a was chosen. As a criterion a minimum of energy at the output of Zero-Forcing-Filter was used[2]. This first step allowed for a more precise estimation of the time constant, which gave a more accurate estimation of the optimum integrator length. Such an iterative process was repeated up to 3 - 4 times. In the simulation carried out for the input signal-to-noise ratio of 20 dB the maximum errors in estimating the time constant 1/a and the amplitude of an exponential signal, did not exceed 10%., Believeably, better accuracy of exponent parameters estimation could be obtained for a more complex (and a more time consuming) identification algorithm.

REFERENCES

1. Dyka, A. : "Liniowa filtracja rozplotowa wzgl dem sygnałów o skończonym czasie trwania", Zesz. Nauk. Politechniki Gdańskiej, Seria Elektronika , Zesz. LXVII, nr 444, Gdańsk 1989, rozprawa habilitacyjna.

2. Dyka, A., Janke, W. : "A Method Of Deconvolution Filtering For Exponential Signals", an unpublished research report.